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Microworld and macroworld - multiscaling problems in modelling of geophysics

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The enormous and the minute are interchangable manifestations of the eternal

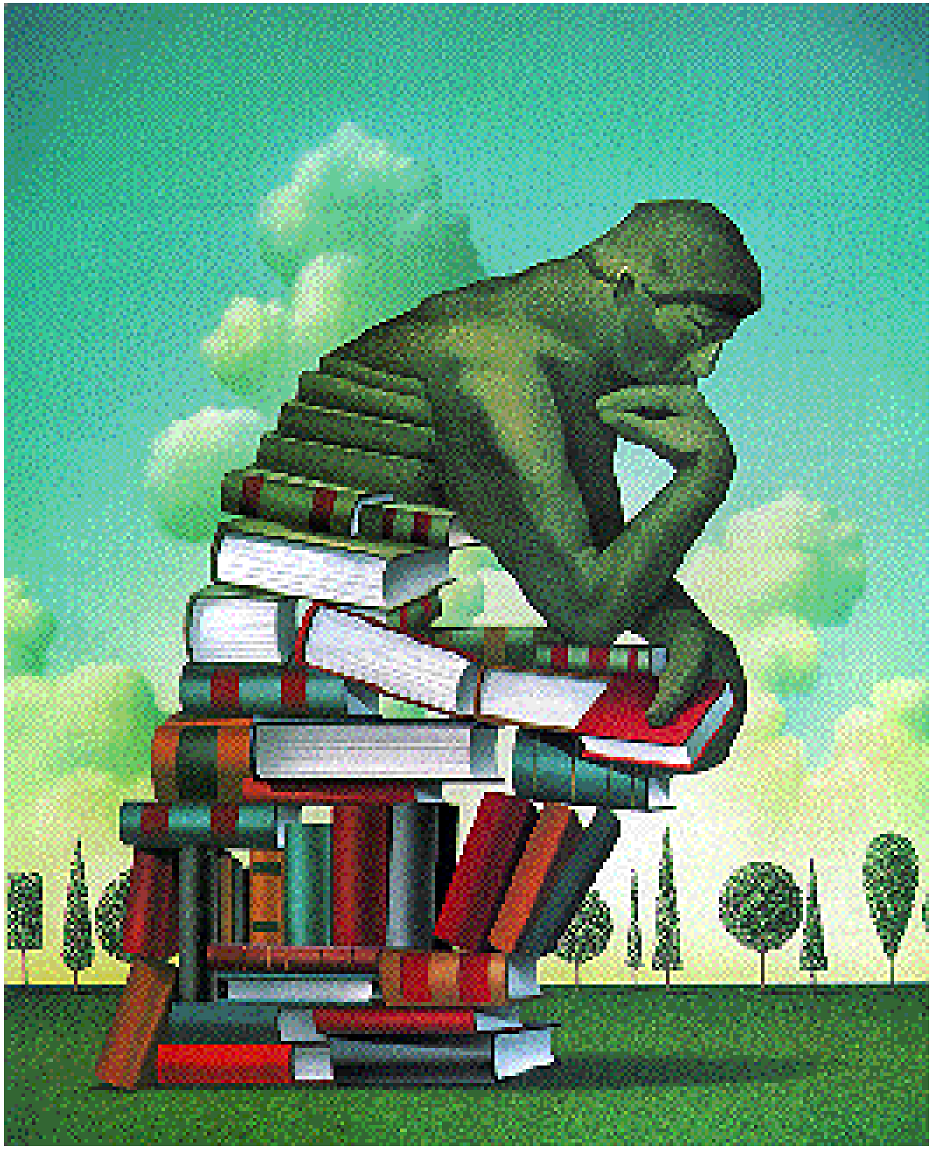
William Blake (1757 – 1827)

The Parable of the Wise and Foolish Virgins

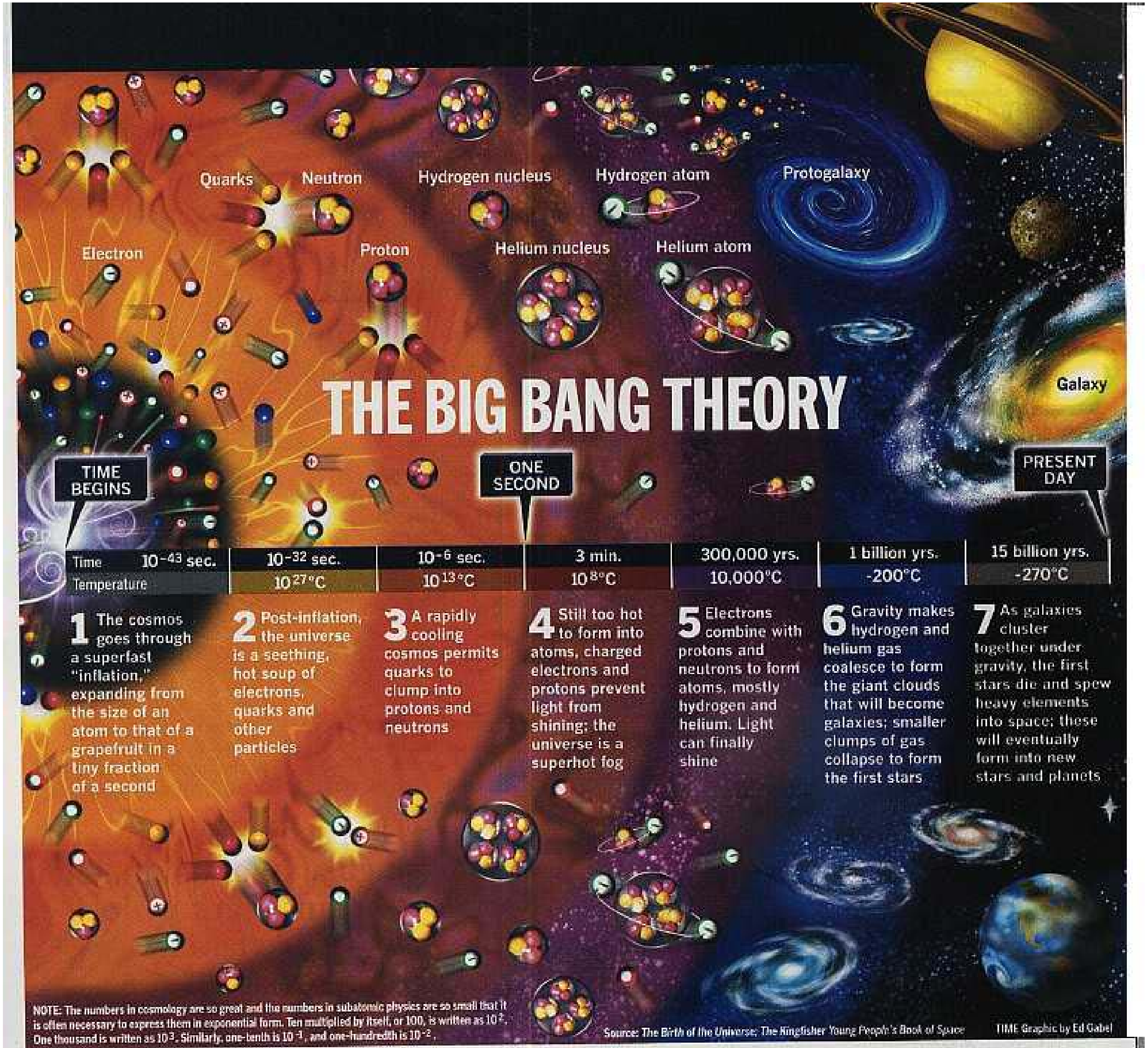
To see a World in a Grain of Sand And a Heaven in a Wild Flower, Hold Infinity in the palm of your hand And Eternity in an hour“

Contents:

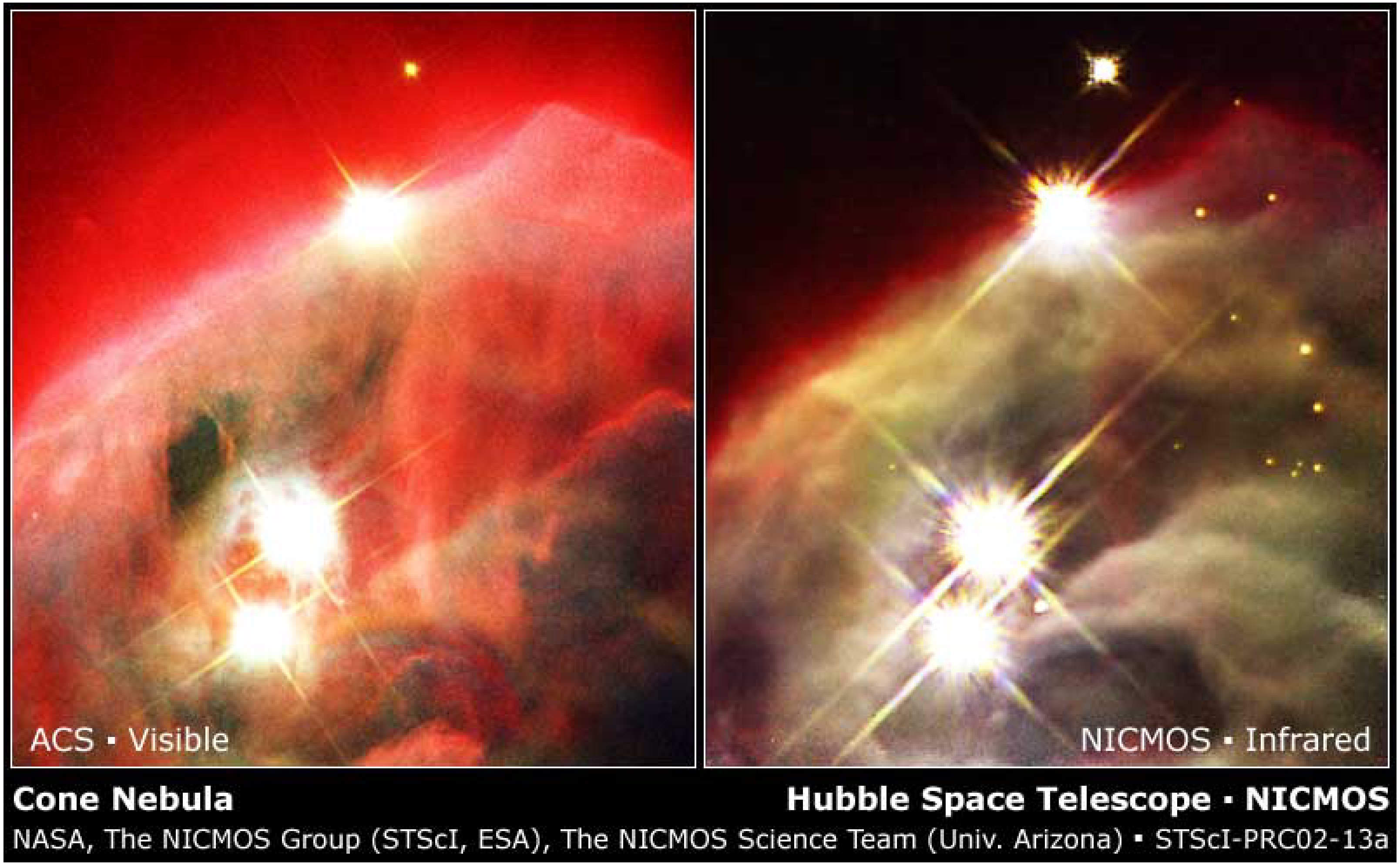
- phenomenology of micro- and macroworld
- scaling of time and space - synchronization and coarse-graining
- from particles to continua in gases -
  BBGKY hierarchy, kinetic theories, extended thermodynamics:
  characteristic relaxation times
- from crystal lattice to continuum mechanics:
  characteristic dimension
- real porous media and thermomechanics of porous
  and granular continua with microstructure:
  characteristic time and length
- example: micro-macrotransition for homogeneous microstructure:
  in situ measurements of porosity
- concluding remarks



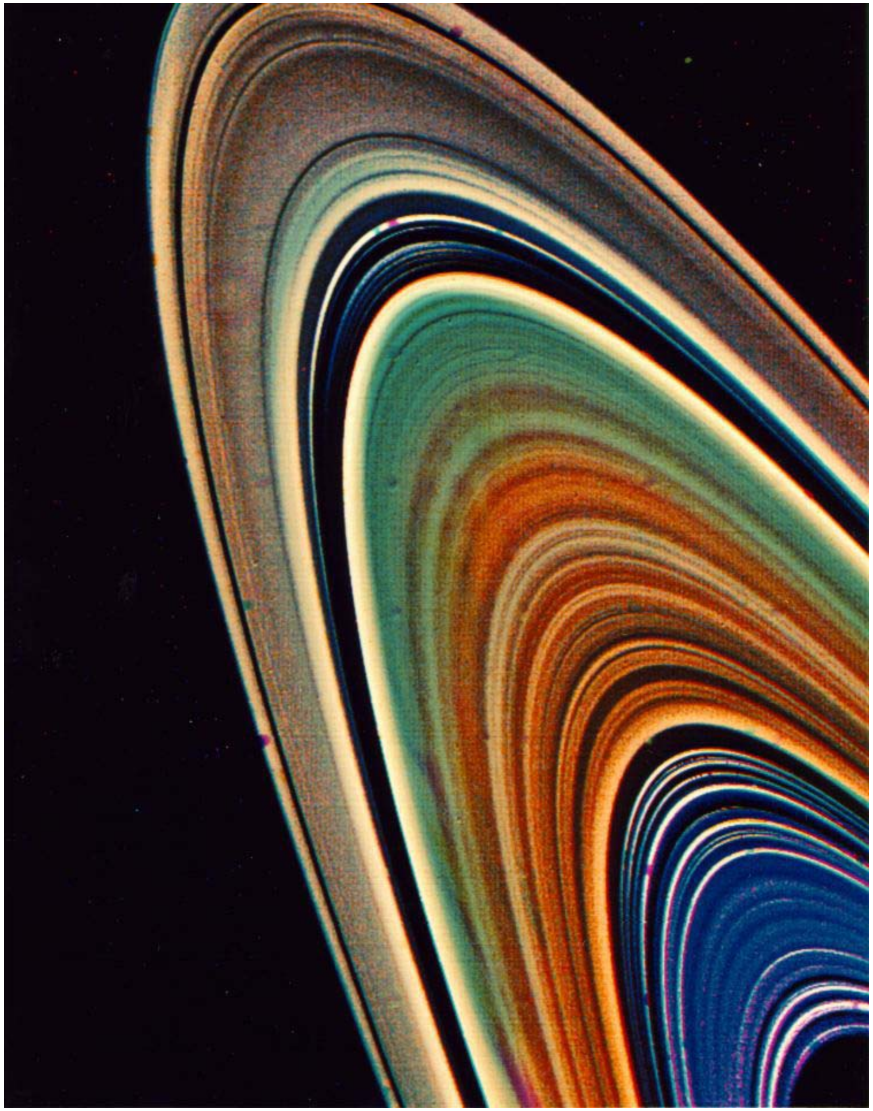
[Phenomenology of micro- and macroworld - a few pictures]



Multiscaling in space-time



Multiscaling in space-time - Boltzmann (relativistic)

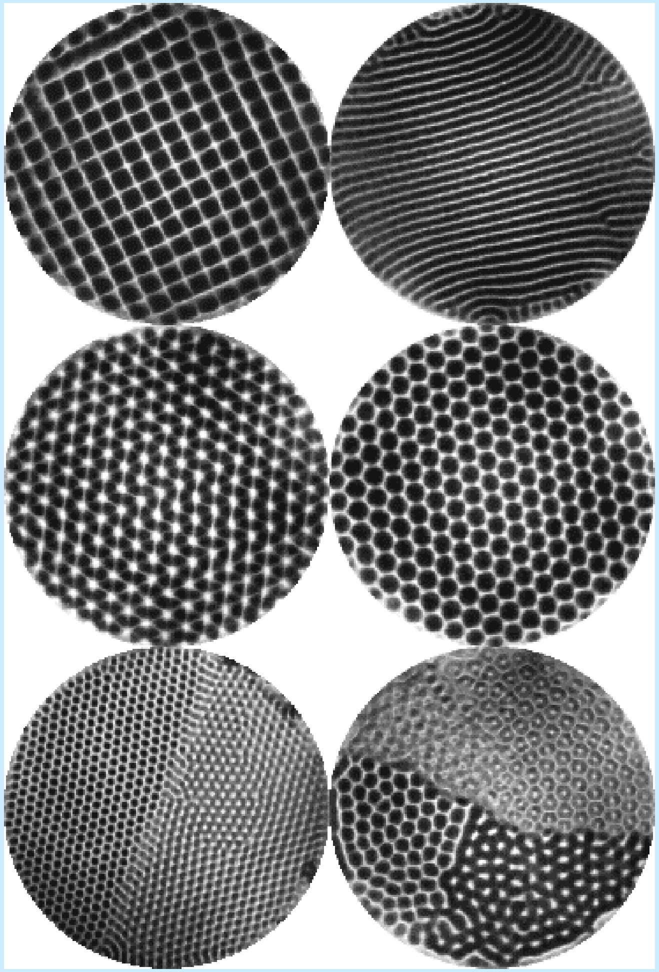


Saturn‘s ring systems - a two-dimensional dry granular system

Thickness ~200 m, mass ~60 g/cm^2, speed <1 mm/s

Multiscaling in space and time - granular gas

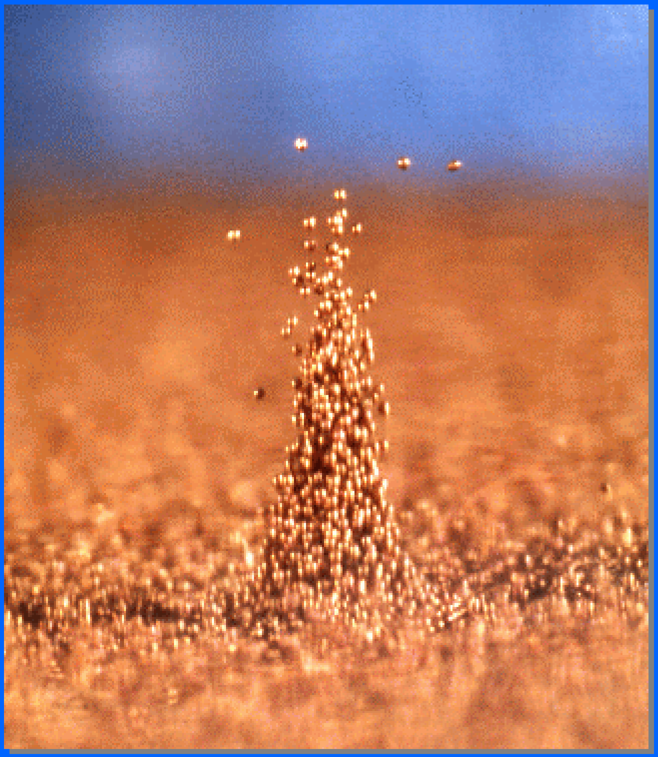
Patterns in vertically oscillated granular layers



Multiscaling in space

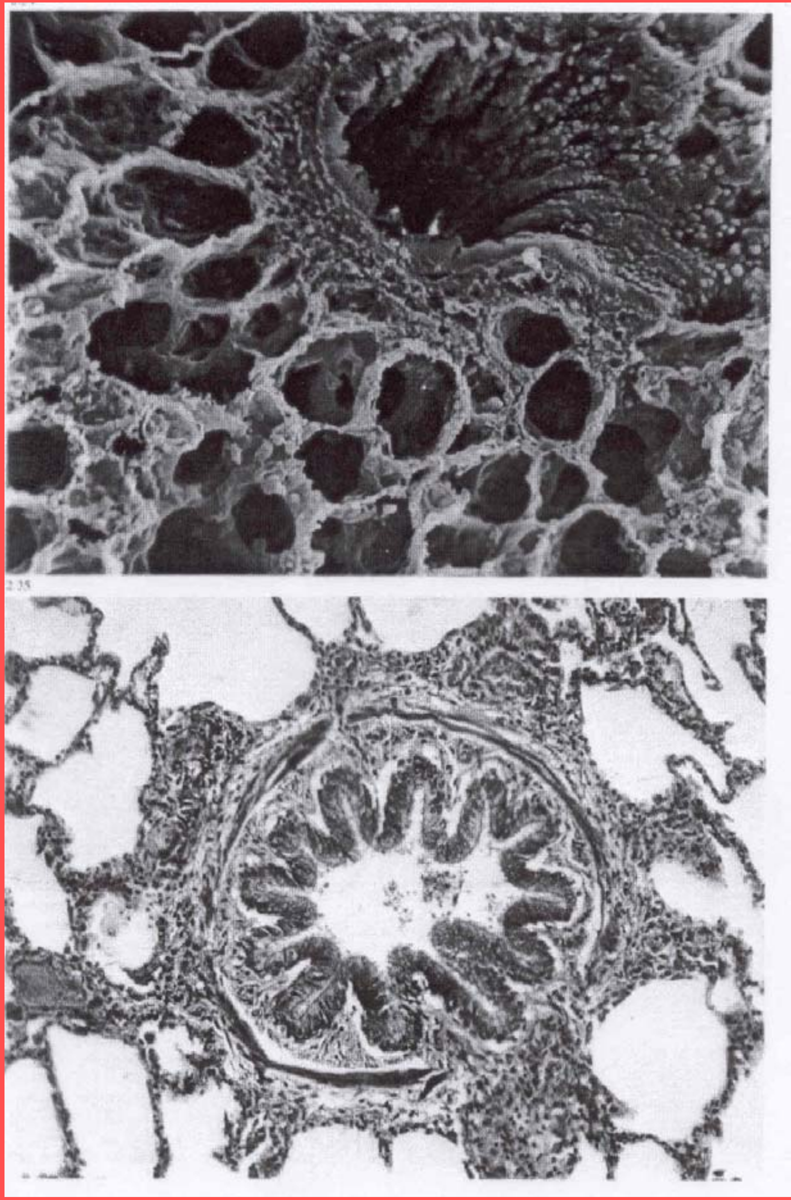
Oscillon

Paul Umbanhowar (Northwestern Univ.)



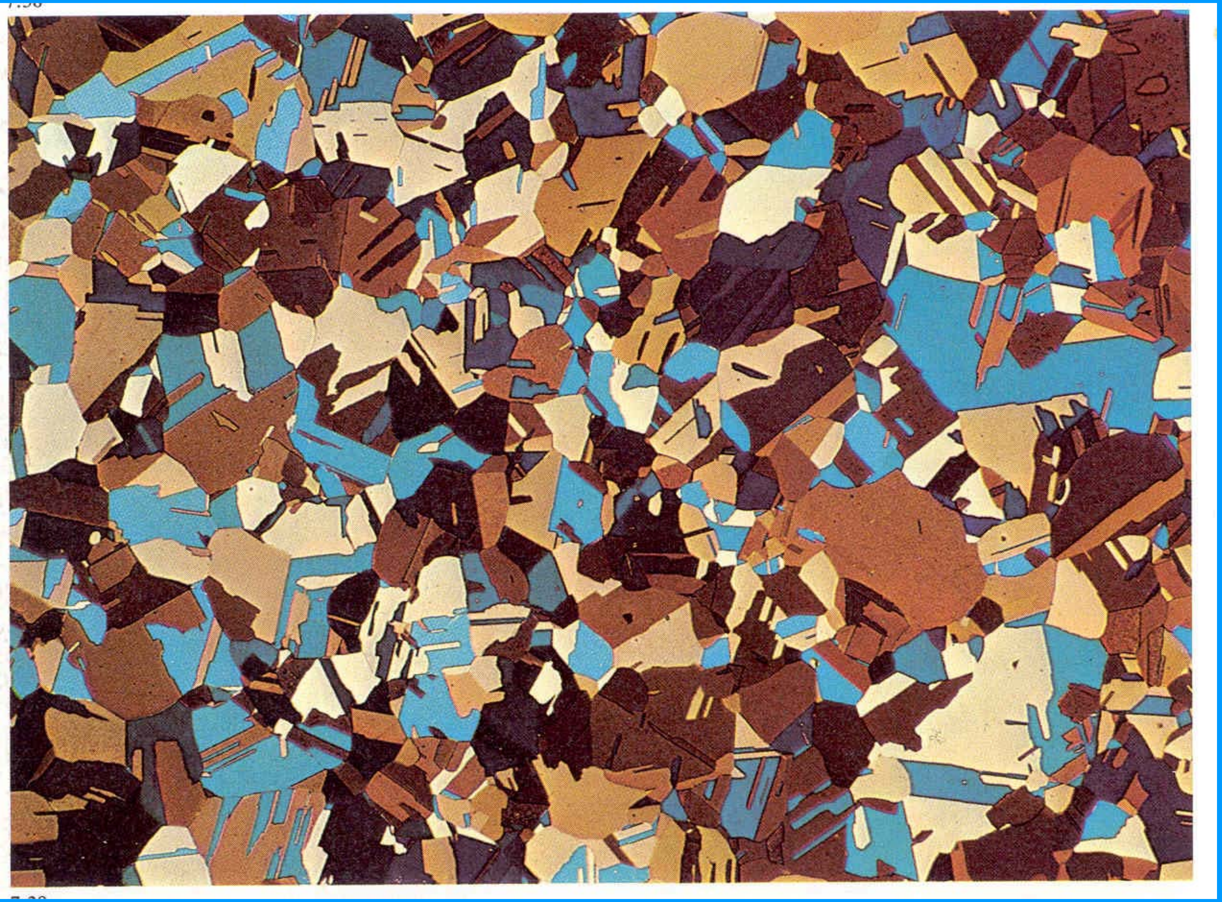
Multiscaling in space and time - granular gas

Alveolae (lungs)



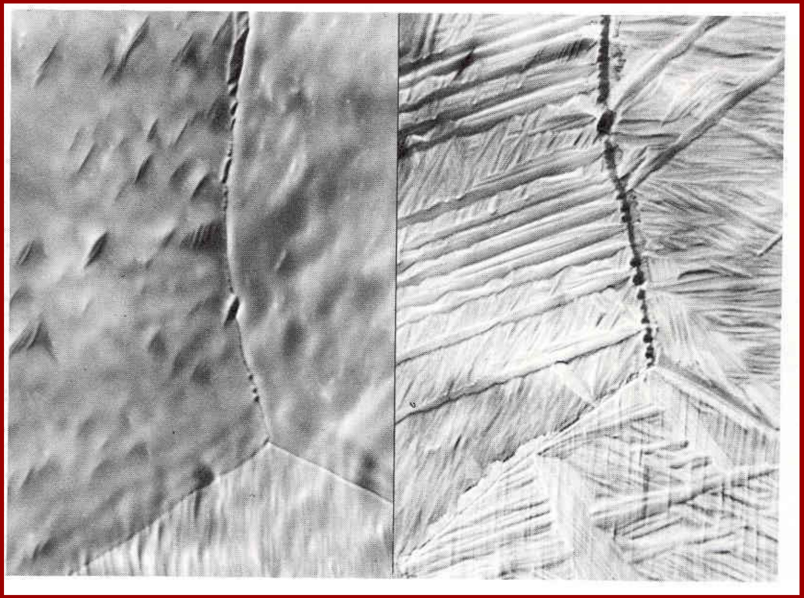
Multiscaling in space

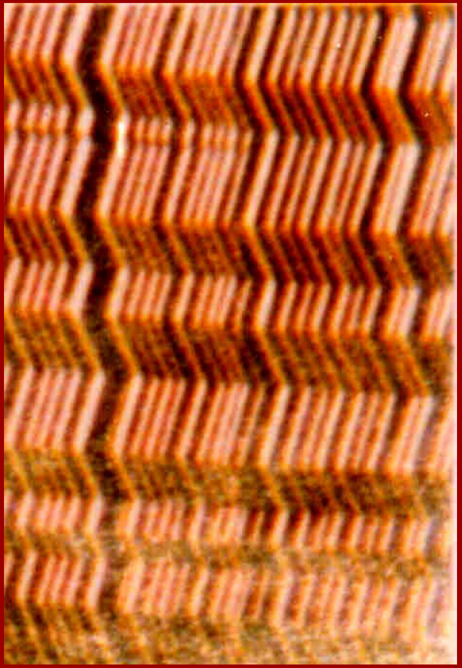
Twinning in steel



Multiscaling in space

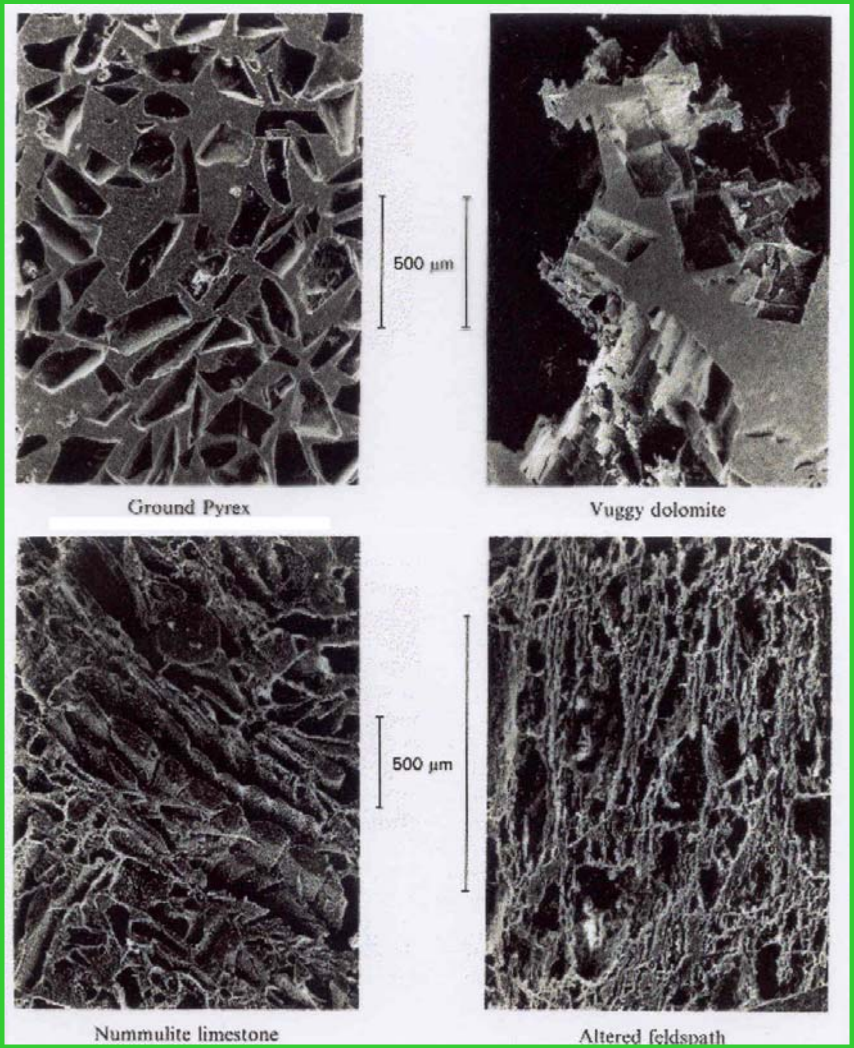
Austenite - martensite phase transformation



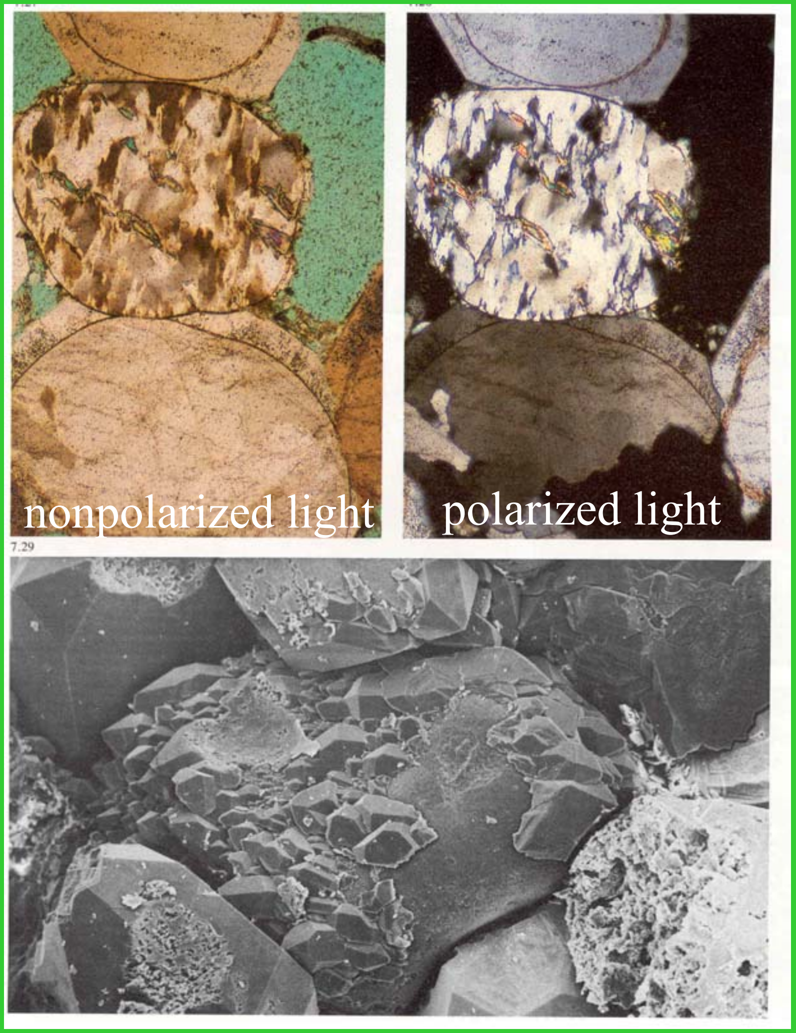


Multiscaling in space

Pore casts (epoxy replicas, Bourbié)



Sandstone in diagenesis



Microscaling in space

Toilet paper

Multiscaling in space

Scaling of time and space - synchronization and coarse-graining for gases

Newton's equations: $\boxed{\dot{\mathbf{y}}_i = \frac{1}{m}\mathbf{p}_i, \quad \dot{\mathbf{p}}_i = \mathbf{F}_i,}$ $\quad \mathbf{F}_i = -\sum_{\substack{j=1,\\ j\neq i}}^{N} \frac{\partial \Phi\left(\left|\mathbf{y}_i - \mathbf{y}_j\right|\right)}{\partial \mathbf{y_i}}.$ (N)

Initial conditions: $\mathbf{y}_i(t=0) = \mathbf{y}_i^0, \quad \dot{\mathbf{y}}_i(t=0) = \dot{\mathbf{y}}_i^0.$ (NIV)

Γ-space: $\Gamma := \{\mathbf{y}_i, \dot{\mathbf{y}}_i; i = 1, \ldots, N\}$ - 6N-dimensional.

Equivalent form - microdistribution function:

$$D(\mathbf{x}_i, \mathbf{v}_i, t) = \prod \delta(\mathbf{x}_i - \mathbf{y}_i(t))\delta(\mathbf{v}_i - \dot{\mathbf{y}}_i(t)).$$

It satisfies the equation (+ initial conditions)

$$\frac{\partial D}{\partial t} + \sum_{i=1}^{N}\left(\mathbf{v}_i \cdot \frac{\partial D}{\partial \mathbf{x}_i} + \frac{1}{m}\mathbf{F}_i \cdot \frac{\partial D}{\partial \mathbf{v}_i}\right) = 0. \qquad \text{(MDE)}$$

Smooth extension:

- in order to smooth out a wild motion of particles (local existence of solutions of (MDE)!) - coarse-graining in Γ-space
- a statistical distribution of initial values (NIV)

A smooth distribution function of N particles satisfies

Joseph Liouville equation (1809 – 1882, eqn. app. 1856)

$$\frac{\partial F^N}{\partial t} + \sum_{i=1}^{N} \left(\mathbf{v}_i \cdot \frac{\partial F^N}{\partial \mathbf{x}_i} + \frac{1}{m} \mathbf{F}_i \cdot \frac{\partial F^N}{\partial \mathbf{v}_i} \right) = 0, \qquad \text{(L)}$$

$$\int_\Gamma F^N d\mathbf{x}_1 \ldots d\mathbf{v}_N = 1, \quad F^N(\mathbf{x}_i, \mathbf{v}_i, t=0) = F_0^N(\mathbf{x}_i, \mathbf{v}_i).$$

From particles to continua for gases - BBGKY hierarchy, kinetic theories, extended thermodynamics

Dimensionless potential: $\Phi\left(\left|\mathbf{x}_i - \mathbf{x}_j\right|\right) = \Phi_0 U_{ij}, \quad \Phi_0 = const.$

$$\boxed{\frac{\partial F^s}{\partial t} + \sum_{i=1}^{s}\left(\mathbf{v}_i \cdot \frac{\partial F^s}{\partial \mathbf{x}_i}\right) - \frac{\Phi_0}{m v_0^2} \sum_{1\le i<j\le s}\left(\frac{\partial U_{ij}}{\partial \mathbf{x}_i}\cdot\frac{\partial F^s}{\partial \mathbf{v}_i} + \frac{\partial U_{ij}}{\partial \mathbf{x}_j}\cdot\frac{\partial F^s}{\partial \mathbf{v}_j}\right)} =$$

$$= \frac{N-s}{N}\left(n r_0^3\right)\left(\frac{\Phi_0}{m v_0^2}\right)\sum_{i=1}^{s}\int \frac{\partial U_{is+1}}{\partial \mathbf{x}_i}\cdot\frac{\partial F^{s+1}}{\partial \mathbf{v}_i}\, d\mathbf{x}_{s+1} d\mathbf{v}_{s+1}, \qquad \text{(BBGKY)}$$

$$F^s := \frac{1}{V^{N-s}}\int F^N d\mathbf{x}_{s+1}\ldots d\mathbf{v}_N, \quad s = 1,\ldots,N,$$

$$\int F^s d\mathbf{x}_1 \ldots d\mathbf{v}_s = \frac{1}{V^{N-s}}, \quad n := \frac{N}{V}.$$

r_0 - range of potential,

v_0 - thermal velocity,

$\theta_0 := \dfrac{r_0}{v_0}$ - time duration of a single collision

N. Bogolubov (1959), M. Born (1949), H. S. Green (1949), I. R. Kirkwood (1935), Yvon (1958)

Characteristic parameters:

$$\frac{\Phi_0}{mv_0^2}$$ - strength of interactions,

$$nr_0^3$$ - number of particles in the range of interaction

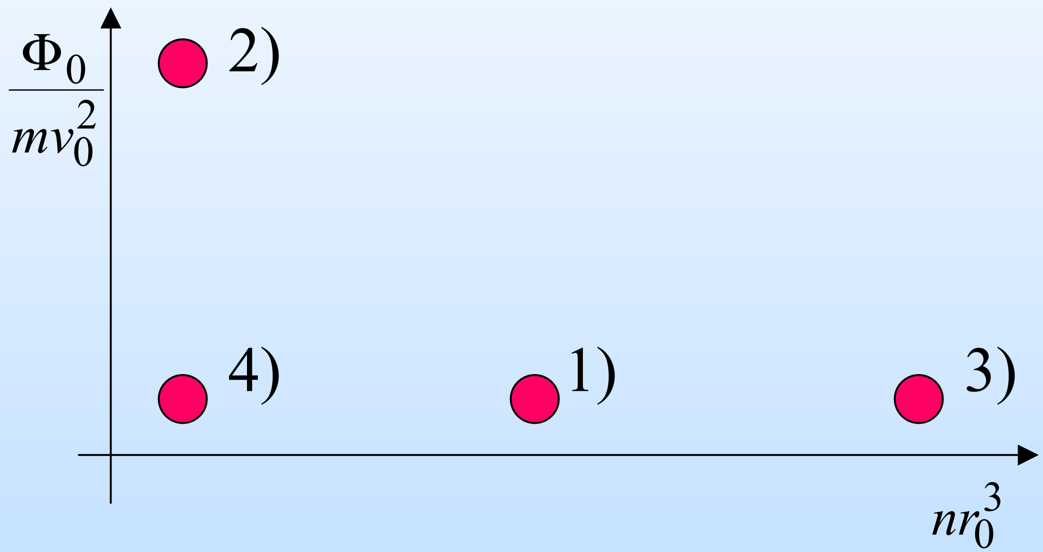
Hence - categories:

1) weakly coupled gas: $nr_0^3 \approx 1, \quad \varepsilon := \dfrac{\Phi_0}{mv_0^2} << 1,$

2) dilute gas with short-range forces: $\varepsilon := nr_0^3 << 1, \quad \dfrac{\Phi_0}{mv_0^2} \approx 1,$

3) gas with Coulomb forces in Debye regime: $nr_0^3 = \dfrac{1}{\varepsilon}, \quad \dfrac{\Phi_0}{mv_0^2} =: \varepsilon << 1,$

4) dilute weakly coupled gas: $nr_0^3 \approx \dfrac{\Phi_0}{mv_0^2} =: \varepsilon << 1.$



Distribution of the models with respect to the parameters

Hence: existence of characteristic relaxation times

$\theta_0 = \frac{r_0}{v_0}$ - time of collision, $t_f = \frac{l_f}{v_0}$ - mean free time,

$t_0 = \frac{L_{macro}}{c_{sound}}$ - macroscopic time, t_p - Poincaré period.

Φ appears in the first equation of the BBGKY-hierarchy only on the right hand side and the coefficient nr_0^3 makes it small. Thus in time intervals θ_0 F^1 changes little while all $F^s, s \geq 2$ change a great deal. This is the collision time scale.

Bogolubov: for sufficiently large times distribution functions $F^s, s \geq 2$ depend on time solely through F^1

$$F^s(\mathbf{x}_1, \ldots, \mathbf{v}_s, t) = F^s\left(\mathbf{x}_1, \ldots, \mathbf{v}_s \middle| F^1(t)\right), s \geq 2$$

Boltzmann kinetic equation

$$\frac{\partial F^1}{\partial t} + \mathbf{v} \cdot \frac{\partial F^1}{\partial \mathbf{x}_1} = \left(nr_0^3\right)\left(\frac{\Phi_0}{mv_0^2}\right)\int \frac{\partial U_{12}}{\partial \mathbf{x}_1} \cdot \frac{\partial F^2}{\partial \mathbf{v}_1} d\mathbf{x}_2 d\mathbf{v}_2. \quad \text{(B)}$$

BGK-approximation

$$\frac{\partial F^1}{\partial t} + \mathbf{v} \cdot \frac{\partial F^1}{\partial \mathbf{x}_1} = -\frac{1}{\theta_0}\left(F^1 - F^1_{equil}\right), \quad \theta_0 \text{ - relaxation time.}$$

Macroscopic theories - moments of one-point distribution function and extended thermodynamics

Generic moments of the one-point distribution function:

$$F_{k_1 k_2 \ldots k_n} = \int m v_{k_1} v_{k_2} \ldots v_{k_n} F^1 d\mathbf{v} \quad \Rightarrow$$

mass density $\quad F = \rho,$

momentum density $\quad F_k = \rho \dot{u}_k, \quad k = 1,2,3,$

momentum flux $\quad F_{kl} = -t_{kl} + \rho \dot{u}_k \dot{u}_l,$

energy density $\quad \frac{1}{2} F_{kk} = \rho \left(\varepsilon + \frac{1}{2} \dot{u}_k \dot{u}_k \right),$

energy flux $\quad \frac{1}{2} F_{kll} = q_k + \rho \left(\varepsilon + \frac{1}{2} \dot{u}_l \dot{u}_l \right) \dot{u}_k - t_{kl} \dot{u}_l,$

Result: balance equations, closure problem - the second law of thermodynamics.

Structure of the macroscopic model in extended thermodynamics

$$\frac{\partial \mathbf{F}_0}{\partial t} + \frac{\partial \mathbf{F}_k}{\partial x_k} = \mathbf{G}, \quad \mathbf{F}_0 = \mathbf{F}_0(\mathbf{u}) \in \Re^n, \quad \mathbf{u} \in \Re^n,$$

$$\mathbf{F}_k = \mathbf{F}_k(\mathbf{u}) \in \Re^n, \quad \mathbf{G} = \mathbf{G}(\mathbf{u}) \in \Re^n, \qquad \text{(FE)}$$

where **u** is the unknown field vector. All solutions of (FE) must fulfil the second law of thermodynamics

$$\frac{\partial h_0}{\partial t} + \frac{\partial h_k}{\partial x_k} \geq 0, \quad h_0 = h_0(\mathbf{u}), \quad h_k = h_k(\mathbf{u}), \qquad \text{(SL)}$$

and sources **G** are zero for conservation quantities. Otherwise they relax to zero in thermodynamical equilibria.

Equations for microstructural variables may be evolutionary, i.e. the corresponding components of $\mathbf{F}_k$ are zero. Then time evolution to equilibrium - scaling with respect to the relaxation time.

Kinetic equation for granular gases

In contrast to classical gases forces between particles are **dissipative**: the force acting on a particle i has the follwing structure

$$m\mathbf{f}_{\mathrm{i}} = \sum_{\mathrm{j}\neq\mathrm{i}} \left(\mathbf{F}_{\mathrm{ij}}^{\mathrm{C}} + \mathbf{F}_{\mathrm{ij}}^{\mathrm{D}}\right),$$

where the conservative and dissipative part are (e.g. T. van Noije, M. H. Ernst (2001))

$$\mathbf{F}_{ij}^{\mathrm{C}} = Y\left(\sigma - r_{ij}\right)\upsilon\left(\sigma - r_{ij}\right)\hat{\mathbf{r}}_{ij}, \quad \mathbf{F}_{ij}^{\mathrm{D}} = -\gamma_n\left(\mathbf{v}_i - \mathbf{v}_j\right)\cdot\hat{\mathbf{r}}_{ij}\upsilon\left(\sigma - r_{ij}\right)\hat{\mathbf{r}}_{ij},$$

$$r_{ij} := \sqrt{(\mathbf{r}_i - \mathbf{r}_j)\cdot(\mathbf{r}_i - \mathbf{r}_j)}, \quad \hat{\mathbf{r}}_{ij} := \frac{\mathbf{r}_i - \mathbf{r}_j}{r_{ij}},$$

and Y is the elasticity of spheres, γ_{n} – coefficient of normal friction (restitution),

υ denotes the charcteristic function, σ is the diameter of the sphere.

Boltzmann equation

$$\frac{\partial F^1}{\partial t} + \mathbf{v}_1 \cdot \frac{\partial F^1}{\partial \mathbf{r}_1} = \frac{1}{m}\frac{\partial}{\partial \mathbf{v}_1}\int d\mathbf{v}_2 \int d\mathbf{r}_{12}\upsilon(\sigma - r_{12})\hat{\mathbf{r}}_{12}\left[\gamma_n(\mathbf{v}_1 - \mathbf{v}_2)\cdot\hat{\mathbf{r}}_{12} - Y\left(\sigma - r_{ij}\right)\right]F^2,$$

and F^2 factorizes.

Results: scarce and solely for quasistatic processes e.g. J. T. Jenkins, I. Goldhirsch (1998), etc.).

Examples of microstructural equations in geophysics

- R. Bowen (1982): evolution equation for volume fractions,

- M. A. Goodman, S. C. Cowin (1972) modified by K. Hutter, B. Svendsen, Y. Wang (1996), (1999): equilibrated force balance

$$\rho k\ddot{\nu} - div\mathbf{h} - \rho f = 0,$$

where ν - volume fraction of solid phase, k - coefficient of equilibrated inertia, **h** and f - equilibrated stress vector and intrinsic equilibrated body force

- K. Wilmanski (1996): balance equation for porosity

$$\dot{n} + div\left[n_E\left(\mathbf{v}^F - \mathbf{v}^S\right)\right] = -\frac{n - n_E}{\tau}, \quad n_E = n_E\left(\frac{\rho_t^F}{\rho_t^S}\right),$$

where n - porosity, $\mathbf{v}^F$, $\mathbf{v}^S$ - partial velocities, ρ_t^F, ρ_t^S - current partial mass densities.

Examples of microstructural equations in geophysics, cont.

• Prandtl – Reuss equation for small elasto-plastic deformations (microstructural variable – plastic deformation)

$$\dot{\mathbf{e}} = \mathbf{C}\dot{\mathbf{T}} + \lambda \frac{\partial \Phi}{\partial \mathbf{T}},$$

where **e** – Almansi-Hamel deformation tensor, **T** – Cauchy stress, **C** – compliance,

• D. Kolymbas (1977): hypoplasticity of granular materials with 4 material parameters C_1, C_2, C_3, C_4 (evolution equation for stresses)

$$\dot{\mathbf{T}} = C_1 (tr\mathbf{T})\mathbf{D} + C_2 \frac{tr(\mathbf{TD})}{tr\mathbf{T}}\mathbf{T} + C_3 \frac{\mathbf{T}^2}{tr\mathbf{T}}\sqrt{tr\mathbf{D}^2} + C_4 \frac{\mathbf{T}_{dev}^2}{tr\mathbf{T}}\sqrt{tr\mathbf{D}^2},$$

where **D** is the rate of logarithmic strain, **T** – Cauchy stress.

The first order equations can be incorporated in ET-structure!

From crystal lattice to continuum mechanics

- the most prominent example for multiscaling in space.

Ergodicity in time averaging („time upscaling“): equivalence of ensemble and time averages

$$\langle p(t)\rangle_{ensemble} = \frac{1}{N}\sum_{n=1}^{N} p_n(t), \quad \langle p(t)\rangle_{time} = \frac{1}{T}\int_0^T p(t-s)ds. \qquad \text{(T)}$$

„Ergodicity“ in space averaging („space upscaling“): equivalence of ensemble and space averages

$$\langle p(\mathbf{x})\rangle_{ensemble} = \frac{1}{N}\sum_{n=1}^{N} p_n(\mathbf{x}), \quad \langle p(\mathbf{x})\rangle_{space} = \frac{1}{V}\int_{REV} p(\mathbf{x}+\mathbf{z})d\mathbf{z}, \qquad \text{(S)}$$

If there is a time relaxation then (T) is plausible for large T; there is no space relaxation. Hence (S) is not very plausible. There exist systems in which space differentiation and REV averages do not comute as they do with ensemble averages (e.g. in wave scattering theory). Application: effective material parameters!

Kröner, Hasihin, Shtrikman, Duvaut, etc.

Real porous media and thermomechanics of porous and granular continua with microstructure

Transition from the pore level of real porous materials to the macroscopic level (upscaling) can be performed by means of at least three methods:

- homogenization,
- averaging over a representative elementary volume (REV),
- averaging over an ensemble.

All require the existence of a characteristic length of microstructure.

We consider solely REV averaging.

For REV: $V(REV) << L^3$, where L is the characteristic macroscopic length.

Result: macroscopic one-component or multicomponent (mixture) models

Examples of mixture models: Biot, Goodman, Cowin, Bowen, Hutter, Svendsen, Wilmanski.

Then macroscopic and microscopic mass densities and momenta of a two-component porous medium are related as follows

$$\rho_t^F(\mathbf{x},t) = \frac{1}{V(REV)} \int_{REV} \rho_t^{FR}(\mathbf{x}+\mathbf{z},t) H^F(\mathbf{x}+\mathbf{z},t) d\mathbf{z},$$

$$\rho_t^S(\mathbf{x},t) = \frac{1}{V(REV)} \int_{REV} \rho_t^{SR}(\mathbf{x}+\mathbf{z},t)\left(1 - H^F(\mathbf{x}+\mathbf{z},t)\right) d\mathbf{z},$$

$$\rho_t^F \mathbf{v}^F(\mathbf{x},t) = \frac{1}{V(REV)} \int_{REV} \rho_t^{FR} \mathbf{v}^{FR}(\mathbf{x}+\mathbf{z},t) H^F(\mathbf{x}+\mathbf{z},t) d\mathbf{z},$$

$$\rho_t^S \mathbf{v}^S(\mathbf{x},t) = \frac{1}{V(REV)} \int_{REV} \rho_t^{SR} \mathbf{v}^{SR}(\mathbf{x}+\mathbf{z},t)\left(1 - H^F(\mathbf{x}+\mathbf{z},t)\right) H^F(\mathbf{x}+\mathbf{z},t) d\mathbf{z},$$

where index t denotes the current configuration and H^F is the characteristic function for the microdomain of the fluid (pore spaces).

Solely in exceptional cases constitutive relations on the level of microstructure can be transferred to the macrolevel!

Example of the macroscopic model with microstructure: poroelastic saturated materials

Balance equations

$$\frac{\partial \rho_t^F}{\partial t} + div\left(\rho_t^F \mathbf{v}^F\right) = 0, \quad \frac{\partial \rho_t^S}{\partial t} + div\left(\rho_t^S \mathbf{v}^S\right) = 0,$$

$$\frac{\partial \rho_t^F \mathbf{v}^F}{\partial t} + div\left(\rho_t^F \mathbf{v}^F \otimes \mathbf{v}^F - \mathbf{T}^F\right) = \hat{\mathbf{p}} + \rho_t^F \mathbf{b}^F,$$

$$\frac{\partial \rho_t^S \mathbf{v}^S}{\partial t} + div\left(\rho_t^S \mathbf{v}^S \otimes \mathbf{v}^S - \mathbf{T}^S\right) = -\hat{\mathbf{p}} + \rho_t^S \mathbf{b}^S,$$

$$\frac{\partial n}{\partial t} + \mathbf{v}^S \cdot grad\, n + div\left[n_E (\mathbf{v}^F - \mathbf{v}^S\right] = \hat{n},$$

Constitutive relations

$$\mathbf{T}^F = -p^F\left(\rho_t^F, n\right), \quad \mathbf{T}^S = \mathbf{T}^S\left(\mathbf{e}^S, n\right), \quad \hat{\mathbf{p}} = -\pi\left(\mathbf{v}^F - \mathbf{v}^S\right),$$

$$\hat{n} = -\frac{n - n_E}{\tau}, \quad n_E = n_0 \frac{\rho_t^F}{\rho_0^F} \frac{\rho_0^S}{\rho_t^S},$$

where the macroscopic form of equations follows from upscaling in space and the evolution of porosity with the relaxation time τ - from the multiscaling of time.

Linear poroelastic saturated materials

Constitutive relations (Wilmanski; simple materials)

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S (tr\mathbf{e}^S)\mathbf{1} + 2\mathrm{G}^S \mathbf{e}^S + \beta(n - n_E)\mathbf{1},$$

$$\mathbf{T}^F = -p_0^F \mathbf{1} - \kappa(\rho^F - \rho_0^F)\mathbf{1} - \beta(n - n_E)\mathbf{1},$$

$$\hat{\mathbf{p}} = -\pi(\mathbf{v}^F - \mathbf{v}^S),$$

Biot‘s constitutive relations (second order material):

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S (tr\mathbf{e}^S)\mathbf{1} + 2\mathrm{G}^S \mathbf{e}^S + n_0 N \zeta \mathbf{1},$$

$$\mathbf{T}^F = -p_0^F \mathbf{1} - R\zeta\mathbf{1} + n_0 N (tr\mathbf{e}^S)\mathbf{1}, \quad \zeta := \frac{n_E - n_0}{n_0},$$

$$\hat{\mathbf{p}} = n_0 N \, grad\, \zeta - \pi(\mathbf{v}^F - \mathbf{v}^S), \quad n \equiv n_E.$$

Example: **Micro-macrotransition for granular materials, homogeneous microstucture**

Macroscopic model: two-component, elastic, Biot-type

Assumptions

- Microstructure is homogeneous within REV; volume REV<<macrovolume
- transition is defined by volumetric Gedankenexperiments with control of pressure (shear modulus is not modelled); Gedankenexperiments are possible in reality
- material consists of two components and REV is material with respect to the skeleton, i.e. the following relation holds

$$\frac{d(\rho^{SR}V^S)}{dt} = 0 \quad \Rightarrow \quad \boxed{\rho^{SR} = \rho_0^{SR}\left(1+e^R\right)^{-1}}$$

where $\rho^{SR}, \rho_0^{SR}, e^R$ denote the current and initial real mass densities of skeleton, and volume change of REV, respectively, $e^R = \dfrac{V^S - V_0^S}{V_0^S}$ where V^S, V_0^S denote volume contributions of the skeleton to REV in current and initial configurations

• volume fraction of the fluid is identical with macroscopic porosity

$$n_0 = \frac{V_0^F}{V_0}, \quad \boxed{n = \frac{V^F}{V}}, \quad V_0^F = V_0 - V_0^S, \quad V^F = V - V^S$$

• porosity is given by changes of macroscopic mass densities

$$\boxed{n = n_0 \frac{\rho_t^F}{\rho_0^F} \frac{\rho_0^S}{\rho_t^S}}$$

• micro and macrodensities are related to each other

$$\boxed{\begin{aligned} &\rho_0^F = n_0 \rho_0^F, \quad \rho_t^F = n \rho^{FR} \\ &\rho_0^S = (1 - n_0) \rho_0^{SR}, \quad \rho_t^S = (1 - n) \rho^{SR} \end{aligned}}$$

• all processes are quasistatic.

Definitions

$$\varepsilon^R = \frac{V^F - V_0^F}{V_0^F}$$ microscopic change of fluid volume

$$\varepsilon = \frac{\rho_0^F}{\rho_t^F} - 1$$ macroscopic change of fluid volume

$$e = J^S - 1,$$ macroscopic change of volume of skeleton

$J^S = 1 + tr\,\mathbf{e}^S$, $\mathbf{e}^S$ - Almansi-Hamel deformation tensor of skeleton

Geometric micro-macrorelations

$$\boxed{\begin{aligned} e &= \frac{1+e^R}{1+\varepsilon^R}\left[n_0\left(1+e^R\right)+\left(1-n_0\right)\left(1+\varepsilon^R\right)\right]-1 \\ \varepsilon &= \frac{1}{1+\varepsilon^R}\left[n_0\left(1+e^R\right)+\left(1-n_0\right)\left(1+\varepsilon^R\right)\right]^2-1 \end{aligned}} \qquad \text{(E1)}$$

micro | macro

Constitutive relations

$$\mathrm{p}^{SR} = -K_S e^R$$
$$\mathrm{p}^{FR} = -K_F \varepsilon^R$$

$$\mathrm{p}^S = -\left(\lambda^S - n_0 N + \frac{2}{3} G^S\right) e - 2 n_0 N \varepsilon$$
$$\mathrm{p}^F = -(R - n_0 N)\varepsilon - 2 n_0 N e \qquad \text{(E2)}$$

where $\mathrm{p}^{SR}, \mathrm{p}^{FR}$ real pressures, $\mathrm{p}^S, \mathrm{p}^F$ partial macropressures

K_S, K_F real bulk modulae, λ^S, N, G^S, R macroscopic elastic parameters

Equilibrium conditions

$$\mathrm{p}' = n_0 \mathrm{p}^{SR} + (1 - n_0)\mathrm{p}^{FR}$$

$$\mathrm{p}' = \mathrm{p}^S + \mathrm{p}^F \qquad \text{(E3)}$$

p' - confining pressure

(E1), (E2) and (E3) yield solution

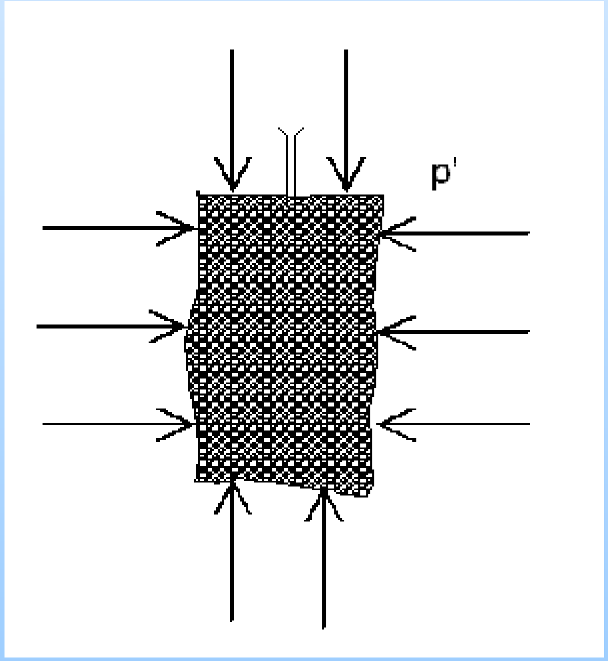
$$\left(\mathrm{p}^S, \mathrm{p}^F, \mathrm{p}^{SR}, \mathrm{p}^{FR}\right)^{\mathrm{T}} = \mathbf{P}\left(\mathrm{p}'; \lambda^{\mathrm{S}}, R, N, K_S, K_F; G^S, n_0\right)$$

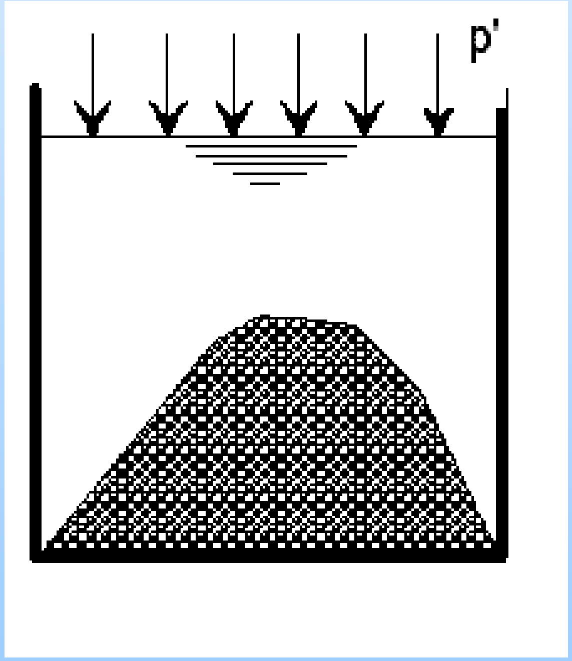
Two additional scalar relations would define two relations between K_S, K_F and $\lambda^S, N, G^S, R, n_0 \quad \Rightarrow$ two Gedankenexperiments

Gedankenexperiments of Biot and Willis

1/ drained jacketed $p^F = 0$

2/ unjacketed $p^{FR} = p'$





Resultant equations

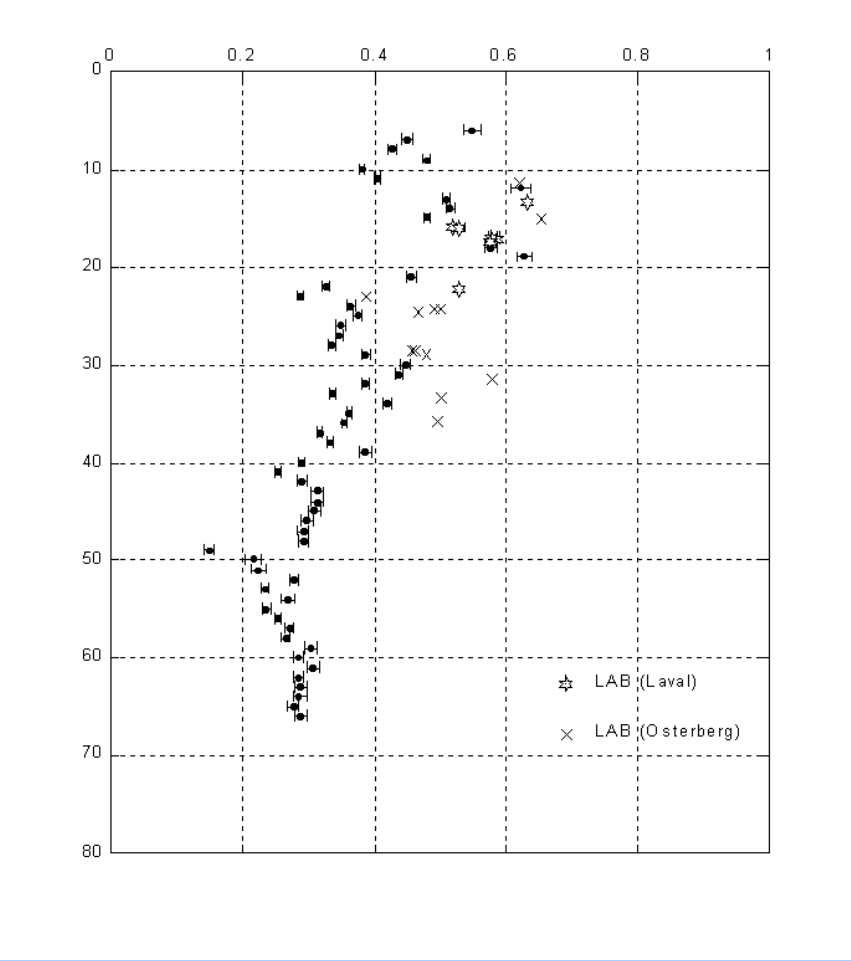
$$\left(\lambda^S+\frac{2}{3}G^S\right)\left(\frac{n_0}{K_F}-\frac{1+n_0}{K_S}\right)+R\left(\frac{n_0}{K_F}-\frac{2n_0}{K_S}\right)+n_0N\left(\frac{2n_0}{K_F}-\frac{1+3n_0}{K_S}\right)+1=0,$$

$$\left(\lambda^S+\frac{2}{3}G^S\right)(R-n_0N)-(R+3n_0N)n_0N+$$
$$+K_F\frac{2n_0^2}{1-n_0}(R+N)-K_S[(1-2n_0)R-n_0N]=0.$$

They relate (K_S, K_F) and (λ^S, R, N) with (G^S, n_0) as parameters

Application: with a given Poisson‘s ratio or a drained compressibility modulus as well as speeds of P1 and S wave one can find the porosity by *in situ* measurements.

Comparison of experimental and theoretical results



Porosity predicted at Pisa site for measured speeds of bulk waves.
Comparison with data from Laval and Osterberg laboratories (C. G. Lai)

Concluding remarks

Example on scaling of independent variables

Example: telegraph equation $$\frac{\partial^2 u}{\partial t^2} + \mu \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0;$$ -hyperbolic (wave) eqn. with damping.

a) to expose vibrations and damping: $\tau = t, \quad \xi = \varepsilon x, \quad \varepsilon << 1.$ Then

$$\frac{\partial^2 u}{\partial \tau^2} + \mu \frac{\partial u}{\partial \tau} \cancel{- c^2 \varepsilon^2 \frac{\partial^2 u}{\partial \xi^2}} = 0;$$ - ordinary differential equation w.r.t. time.

b) to expose static deformations: $\tau = \varepsilon t, \quad \xi = x, \quad \varepsilon << 1.$ Then

$$\cancel{\varepsilon^2 \frac{\partial^2 u}{\partial t^2} + \mu\varepsilon \frac{\partial u}{\partial t}} - c^2 \frac{\partial^2 u}{\partial x^2} = 0;$$ - ordinary differential equation w.r.t. space.

c) to expose diffusion: $\tau = \varepsilon t, \quad \xi = \sqrt{\varepsilon} x, \quad \varepsilon << 1.$ Then

$$\cancel{\varepsilon^2 \frac{\partial^2 u}{\partial \tau^2}} + \mu\varepsilon \frac{\partial u}{\partial \tau} - c^2 \varepsilon \frac{\partial^2 u}{\partial \xi^2} = 0;$$ - parabolic (diffusion) equation.

Geophysical application: Partial momentum balance for fluid vs. Darcy's law

hyperbolic

parabolic

Concluding remarks

On field multiscaling

1. Multiscaling in time yields a possibility to construct a hierarchy of fields which relax one after another to a macroscopic thermodynamical equilibrium. Dynamics of the last few steps in the hierarchy can be reflected by a time synchronization. Consequences: kinetic regime, thermodynamical regime, ergodicity.

2. In contrast to the theory of ideal gases (BBGKY) modeling by means of time multiscaling for granular materials has not been performed for the whole hierarchy. In the kinetic regime one has to introduce a dissipation in the microscopic range (friction and energy restitution). The classical H-theorem of Boltzmann‘s theory – a precursor of the second law of thermodynamics – does not hold.

3. Multiscaling in space yields the existence of microstructure on which a hierarchy of fields can be constructued. They do not have to relax (some do – e.g. dynamical changes of porosity described by evolution equations). Different averaging procedures – homogenization, space averaging in REV, ensemble averages – do not have to be equivalent.

4. Multiscaling in space for porous materials should yield the existence of such additional fields as a local curvature of channels (tortuosity), micro-structural anisotropy (a tensor of permeability), influence of corners, micro-vorticities, creation of large gradients of porosity (liquefaction of sands), etc. These problems have not been yet addressed in continuum modeling.