Pieter Brueghel the Elder (1525 – 1569), *An alchemist at work*

Leibniz Gemeinschaft

WIAS

Weierstrass Institute for Applied Analysis and Stochastics

in Forschungsverbund Berlin e.V., Mohrenstrasse 39, D - 10117 Berlin, Germany

Critical Time for Acoustic Waves in Weakly Nonlinear Poroelastic Materials

KRZYSZTOF WILMAŃSKI

mail: wilmansk@wias-berlin.de
web: http://www.wias-berlin.de/people/wilmansk

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Liquefaction and other ground instabilities

Ground rupture (Taiwan)

Landslide in El Salvador (Colonia Las Colinas) by the earthquake 13.01.2001

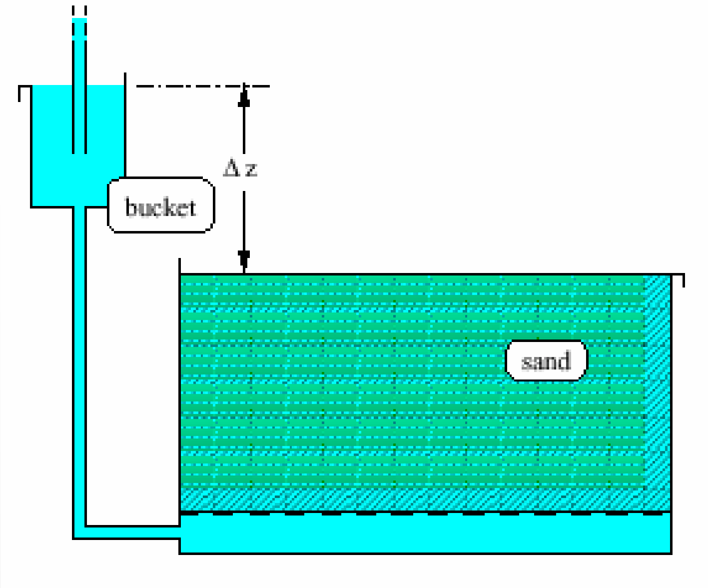
Liquefaction after Niigata earthquake Japan 1964

Tilt due to liquefaction (Adapazari)

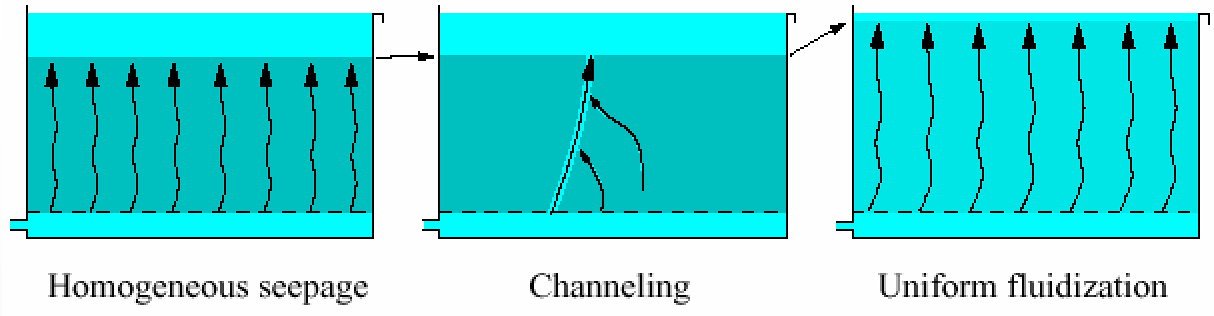
Experiments on a saturated sand

PhD-Thesis: Theo Wilhelm, University of Innsbruck, 2000

Theo Wilhelm, K. Wilmanski; *On the Onset of Flow Instabilities in Granular Media due to Porosity Inhomogeneities, Int. J. Multiphase Flows,* 28, 1929-1944, 2002.

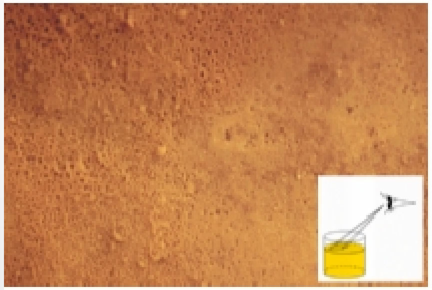


Experimental setup

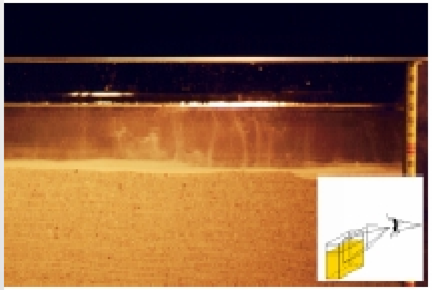


Flow regimes in sand-water mixtures under seepage conditions

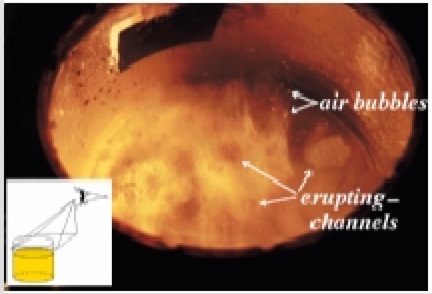
1. Hemogeneously distributed micro-channels (small dark spots) on the top surface of a sand speciment subject to seepage. Diameters of channels up to 1 mm.

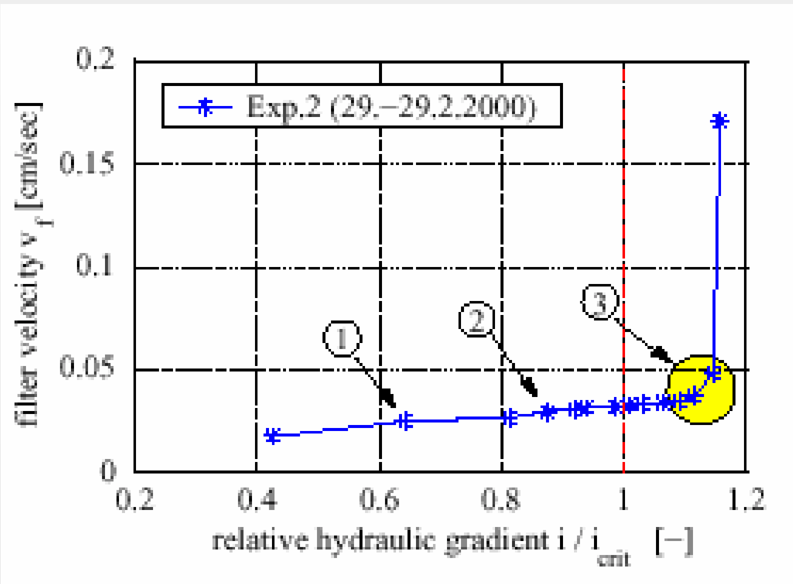


2. Channels with diameters up to se-veral mm have formed. Very fine par-ticles flushed out through them are visible in the water layer above the sand surface.



3. Instabilities (washed out air bubbles, erupting channels) shortly before the eruption of a main channel.





Experimental data from a seepage experiment.

Two-component weakly nonlinear model of poroelastic saturated media

Fields: $\left(\mathbf{x},t\right)\rightarrow\left\{\rho^S,\rho^F,n,\mathbf{v}^S,\mathbf{v}^F,\mathbf{e}^S\right\}.$

ρ^S - partial mass density of the skeleton,

ρ^F - partial density of the fluid,

n - porosity (volume fraction of the fluid in REV),

$\mathbf{v}^S$ - velocity of the skeleton,

$\mathbf{v}^F$ - velocity of the fluid,

$\mathbf{e}^S$ - Almansi-Hamel deformation tensor of the skeleton.

$$\varepsilon=\frac{\rho_0^F-\rho^F}{\rho_0^F}$$ -volume changes of the fluid,

Second order approximation:

$$\max\left\{\sum_{i,j}\left|\lambda_e^{(i)}\lambda_e^{(j)}\right|\right\}<<1,\quad \det\left(\mathbf{e}^S-\lambda_e^{(i)}\mathbf{1}\right)=0,$$

$$\left|\varepsilon\right|<<1.$$

Equilibrium porosity: $n \approx n_E = n_0\left(1 + \delta\, tr\, \mathbf{e}^S\right).$

Mass density of the skeleton: $\rho^S = \rho_0^S\left(1 - I - \frac{1}{2}\left(I^2 + 4II\right)\right), \quad I = tr\mathbf{e}^S, \quad II = \frac{1}{2}\left(I^2 - tr\mathbf{e}^{S2}\right).$

Remaining Field Equations:

$$\frac{\partial \varepsilon}{\partial t} + div\left(\varepsilon - 1\right)\mathbf{v}^F = 0,$$

$$\rho^S\left(\frac{\partial \mathbf{v}^S}{\partial t} + \mathbf{L}^S\mathbf{v}^S\right) = div\,\mathbf{T}^S + \pi\left(\mathbf{v}^F - \mathbf{v}^S\right),$$

$$\rho^F\left(\frac{\partial \mathbf{v}^F}{\partial t} + \mathbf{L}^F\mathbf{v}^F\right) = -grad\, p^F - \pi\left(\mathbf{v}^F - \mathbf{v}^S\right),$$

$$\mathbf{L}^S = grad\,\mathbf{v}^S, \quad \mathbf{L}^F = grad\,\mathbf{v}^F.$$

Integrability condition:

$$\frac{\partial \mathbf{e}^S}{\partial t} + \mathbf{v}^S \cdot grad\,\mathbf{e}^S = \frac{1}{2}\left(\mathbf{L}^S + \mathbf{L}^{ST}\right) - \\ - \left(\mathbf{L}^{ST}\mathbf{e}^S + \mathbf{e}\mathbf{L}^S\right).$$

Constitutive relations (Signorini-like):

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda_0^S I\mathbf{1} + 2\mu_0^S\mathbf{e}^S + \left(\delta \left.\frac{\partial \lambda^S}{\partial n}\right|_0 n_0 I + \frac{1}{2}\left(\lambda_0^S + \mu_0^S\right)I^2\right)\mathbf{1} + \\ + 2\left(\delta \left.\frac{\partial \mu^S}{\partial n}\right|_0 n_0 I - \left(\lambda_0^S + \mu_0^S\right)I\right)\mathbf{e}^S,$$

$$p^F = p_0^F - \rho_0^F \kappa_0 \varepsilon - \rho_0^F \delta \left.\frac{\partial \kappa}{\partial n}\right|_0 n_0 I \varepsilon.$$

1D Model

$$\mathbf{v}^S = v^S \mathbf{e}_x, \quad \mathbf{v}^F = v^F \mathbf{e}_x, \quad \mathbf{e}^S = e^S \mathbf{e}_x \otimes \mathbf{e}_x, \quad |\mathbf{e}_x| = 1.$$

Hence

$$I = e^S, \quad II = 0, \quad \mathbf{L}^S = \frac{\partial v^S}{\partial x}\mathbf{e}_x \otimes \mathbf{e}_x, \quad \mathbf{L}^F = \frac{\partial v^F}{\partial x}\mathbf{e}_x \otimes \mathbf{e}_x.$$

$$\rho^S = \rho_0^S\left(1 - e^S - \frac{1}{2}e^{S2}\right), \quad \rho^F = \rho_0^F(1-\varepsilon), \quad n = n_0(1 + \delta e^S).$$

Partial stresses:

$$\sigma^S = \sigma_0^S + (\lambda^S + 2\mu^S)e^S - \frac{3}{2}(\lambda_0^S + \mu_0^S)e^{S2}, \quad p^F = p_0^F - \rho_0^F \kappa\varepsilon,$$

$$\lambda^S + 2\mu^S = \lambda_0^S + 2\mu_0^S + \delta n_0 \frac{\partial}{\partial n}(\lambda^S + 2\mu^S)\Big|_0 e^S.$$

$$\kappa = \kappa_0 + \delta n_0 \frac{\partial \kappa}{\partial n}\Big|_0 e^S.$$

Governing set of equations:

$$\frac{\partial u'_A}{\partial t'} + A'_{AB}\frac{\partial u'_B}{\partial x'} = B'_A, \quad t' = \frac{t\pi}{2\rho_0^S}, \quad x' = \frac{x\pi}{2\rho_0^S c_{P1}},$$

$$c_{P1}^2 = \frac{\lambda_0^S + 2\mu_0^S}{\rho_0^S}, \quad c_S^2 = \frac{\mu_0^S}{\rho_0^S}, \quad c_{P2}^2 = \kappa_0, \quad c_s = \frac{c_S}{c_{P1}}, \quad c_f = \frac{c_{P2}}{c_{p1}},$$

Auxiliary quantities:

$$l^F{}' = \delta n_0 \frac{\partial}{\partial n} \frac{\rho_0^S \kappa}{\lambda_0^S + 2\mu_0^S}\Bigg|_0,$$

$$l^S{}' = 2\delta n_0 \frac{\partial}{\partial n} \frac{\lambda^S + 2\mu^S}{\lambda_0^S + 2\mu_0^S}\Bigg|_0 - \left(2 - 3c_s^2\right).$$

$$[u'_A]^T = \left[\varepsilon, v^F{}', v^S{}', e^S\right]^T, \quad v^F{}' = \frac{v^F}{c_{P1}}, \quad v^S{}' = \frac{v^S}{c_{P1}},$$

★

$$[A'_{AB}] = \begin{bmatrix} v^F{}' & \varepsilon - 1 & 0 & 0 \\ -c_f^2(1+\varepsilon) - l^F{}'e^S & v^F{}' & 0 & -l^F{}'\varepsilon \\ 0 & 0 & v^S{}' & -1 - l^S{}'e^S \\ 0 & 0 & -\left(1 - 2e^S\right) & v^S{}' \end{bmatrix},$$

$$[B'_A]^T = \left[0, -2(1+\varepsilon)\left(v^F{}' - v^S{}'\right), 2\left(1 + e^S\right)\left(v^F{}' - v^S{}'\right), 0\right]^T.$$

Evolution of the amplitude of weak discontinuity

Wave front S

$$[[u'_A]] = (u'_A)^+ - (u'_A)^-, \quad \left[\left[\frac{\partial u'_A}{\partial t'}\right]\right] = -c\left[\left[\frac{\partial u'_A}{\partial x'}\right]\right],$$

$$\left[\left[\frac{\partial^2 u'_A}{\partial t'\partial x'}\right]\right] = \frac{d}{dt'}\left[\left[\frac{\partial u'_A}{\partial x'}\right]\right] - c\left[\left[\frac{\partial^2 u'_A}{\partial x'^2}\right]\right], \quad etc.,$$

$$(A'_{AB} - c\delta_{AB})\left[\left[\frac{\partial u'_A}{\partial x'}\right]\right] = 0 \quad \Rightarrow \quad \boxed{\left[\left[\frac{\partial u'_A}{\partial x'}\right]\right] = \mathrm{A} r'_A,} \quad r'_A r'_A = 1.$$

Evolution of the amplitude

$$\frac{d\mathrm{A}}{dt'} + \alpha'_1 \mathrm{A} + \alpha'_2 \mathrm{A}^2 = 0 \quad \Rightarrow$$

$$\Rightarrow \quad \frac{1}{\mathrm{A}} = \left[\frac{1}{\mathrm{A}_0} + \int_0^{t'} \alpha'_2 \exp\left(-\int_0^{\eta} \alpha'_1\, ds\right) d\eta\right] \exp\left(\int_0^{t'} \alpha'_1\, ds\right).$$

Critical time

$$\left[\frac{1}{\mathrm{A}_0} + \int_0^{t'_c} \alpha'_2 \exp\left(-\int_0^{\eta} \alpha'_1\, ds\right) d\eta\right] = 0.$$

★ Matrix A'_{AB} on the positive side of the P1-front – solution of the eigenvalue problem

Eigen-value	Right eigenvector r'_A	Left eigenvector l'_A
+1	$[0,0,-1/\sqrt{2},1/\sqrt{2}]$	$[0,0,-1/\sqrt{2},1/\sqrt{2}]$
-1	$[0,0,1/\sqrt{2},1/\sqrt{2}]$	$[0,0,1/\sqrt{2},1/\sqrt{2}]$
c_f	$[1/\sqrt{(1+c_f^2)},-c_f/\sqrt{(1+c_f^2)},0,0]$	$[c_f/\sqrt{(1+c_f^2)},-1/\sqrt{(1+c_f^2)},0,0]$
$-c_f$	$[1/\sqrt{(1+c_f^2)},c_f/\sqrt{(1+c_f^2)},0,0]$	$[c_f/\sqrt{(1+c_f^2)},1/\sqrt{(1+c_f^2)},0,0]$

Coefficients in the equation for the amplitude – P1-characteristic:

$$\alpha'^{(1)}_1 = -l'_A \frac{\partial B'_A}{\partial u'_C} r'_C \frac{1}{r'_D l'_D}\bigg|^{(1)} = 1,$$

$$\alpha'^{(1)}_2 = -l'_A \frac{\partial A'_{AB}}{\partial u'_C} r'_B r'_C \frac{1}{r'_D l'_D}\bigg|^{(1)} = -\frac{1}{\sqrt{2}}\left(2 - \tfrac{1}{2} l^{S}{}'\right).$$

Solution:

$$\mathrm{A} = e^{-t'}\left[\frac{1}{\mathrm{A}_0} - \frac{1}{\sqrt{2}}\left(2 - \tfrac{1}{2} l^{S}{}'\right)\left(1 - e^{-t'}\right)\right]^{-1}.$$

Critical time:

$$t'_c = -\ln\left[1 - \frac{\sqrt{2}}{\mathrm{A}_0\left(2 - \tfrac{1}{2} l^{S}{}'\right)}\right].$$

Threshold amplitude

$$\mathrm{A}_0 > \frac{\sqrt{2}}{2 - \tfrac{1}{2} l^{S}{}'}.$$

Existence of critical time: $\mathrm{A}_0\left(2 - \tfrac{1}{2} l^{S}{}'\right) > 0.$

Physical amplitudes:

$$\left[\left[\frac{\partial e^S}{\partial x'}\right]\right] = -\left.\frac{\partial e^S}{\partial x'}\right|^- = \mathrm{A}r'^{(1)}_4 = \tfrac{1}{\sqrt{2}}\mathrm{A} \quad \Rightarrow \quad \left.\frac{\partial e^S}{\partial x}\right|^- = -\frac{\pi}{2\sqrt{2}\rho_0^S c_{P1}}\mathrm{A}.$$

$$\left[\left[\frac{\partial v'^S}{\partial x'}\right]\right] = -\left.\frac{\partial v'^S}{\partial x'}\right|^- = \mathrm{A}r'^{(1)}_3 = -\tfrac{1}{\sqrt{2}}\mathrm{A} \quad \Rightarrow \quad \left.\frac{\partial e^S}{\partial t}\right|^- = \frac{\pi}{2\sqrt{2}\rho_0^S}\mathrm{A}.$$

Numerical example:

$$\pi = 10^7 \tfrac{kg}{m^3 s} \quad (app.\, 0.1\, Darcy), \quad \rho_0^S = 2500 \tfrac{kg}{m^3}, \quad c_{P1} = 2500 \tfrac{m}{s},$$

$$\mathrm{A} = 0.4 \quad \Rightarrow \quad \left.\frac{\partial e^S}{\partial x}\right|^- \approx 0.25 \tfrac{1}{m}.$$

P2-characteristic is much slower than P1 and enters a disturbed region.
It is not essential for the critical behavior.

Micro-macro; numerical results

Gassmann-type relations - macroparameters in function of porosity:

$$\delta = \frac{K_V - K}{n\left(K_s - K_f\right)}, \quad K = \lambda_0^S + \frac{2}{3}\mu_0^S + \rho_0^F \kappa_0, \quad K_V = (1-n)K_s + nK_f,$$

$$\lambda^S + 2\mu^S = \frac{3(1-\nu)}{1+\nu}\left\{\frac{(K_s - K_d)^2}{\frac{K_s^2}{K_W} - K_d} + K_d\right\}, \quad \frac{1}{K_W} = \frac{1-n}{K_s} + \frac{n}{K_f},$$

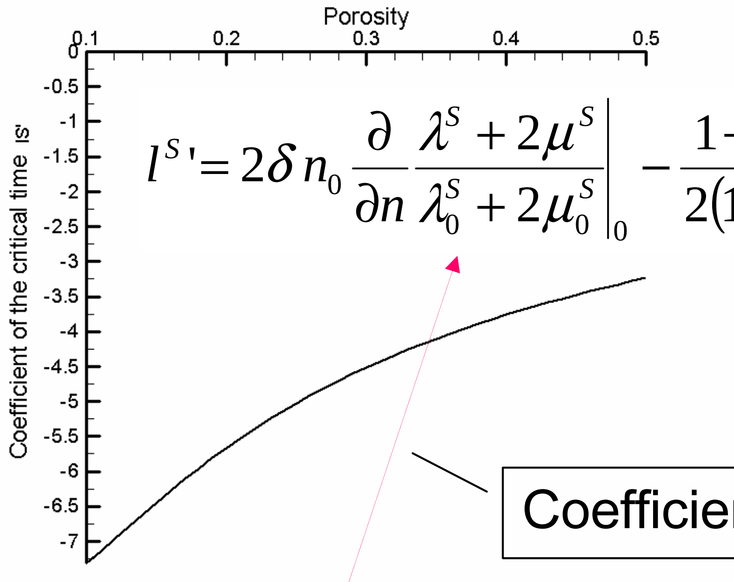
$$\mu^S = \frac{3(1-2\nu)}{2(1+\nu)}\left\{\frac{(K_s - K_d)^2}{\frac{K_s^2}{K_W} - K_d} + K_d\right\},$$

$$\rho_0^F \kappa = n^2 \frac{K_s^2}{\frac{K_s^2}{K_W} - K_d}, \qquad K_d = \frac{K_s}{1+50n}.$$

Geertsma empirical relation

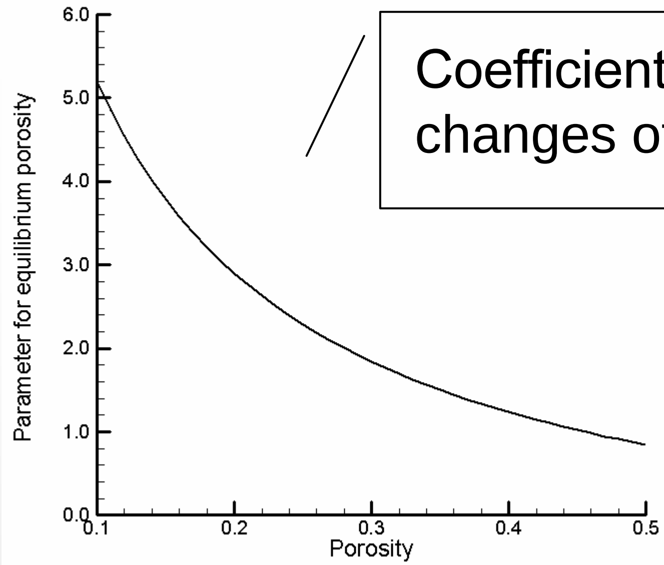
Given:

$$K_s, K_f, \nu$$

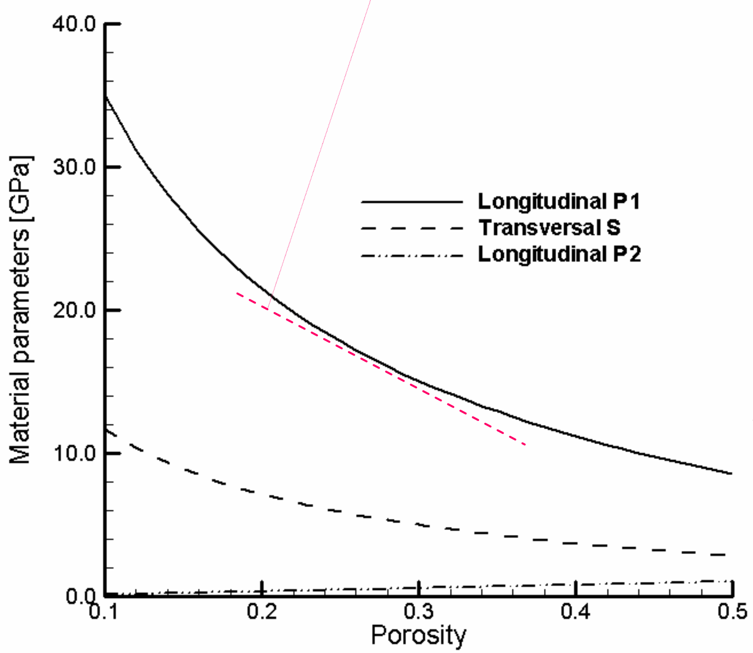


$$l^{S}{}' = 2\delta\, n_0 \frac{\partial}{\partial n} \frac{\lambda^S + 2\mu^S}{\lambda_0^S + 2\mu_0^S}\bigg|_0 - \frac{1+2\nu}{2(1-\nu)} < 0.$$

Coefficient $l^{S}{}'$



Coefficient δ for changes of porosity



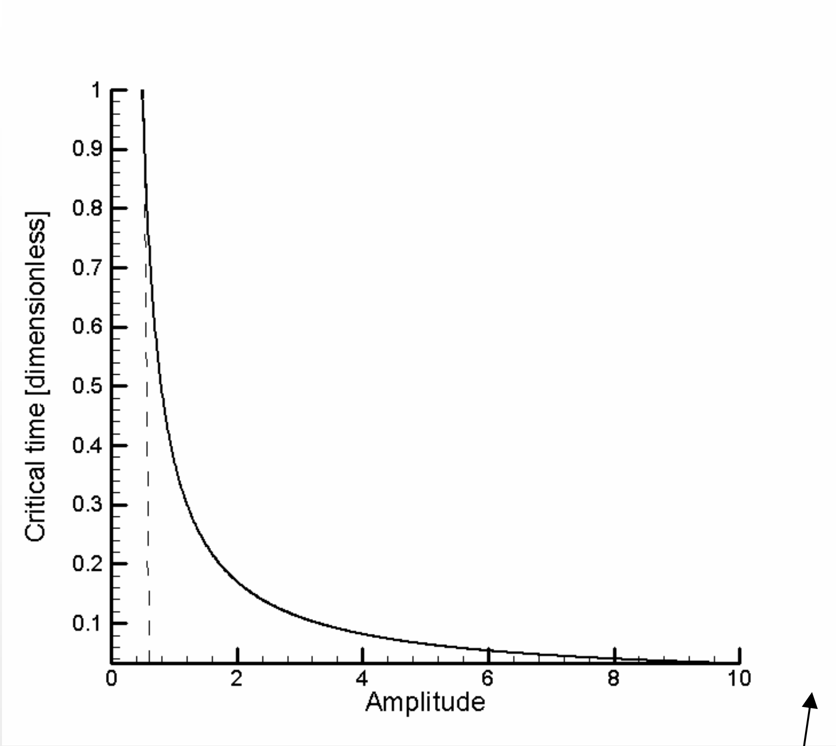
Material parameters:

$$\lambda^S + 2\mu^S, \quad \mu^S, \quad \rho_0^F \kappa$$

Numerical data:

$$K_s = 48 GPa, \quad K_f = 2.25 GPa,$$

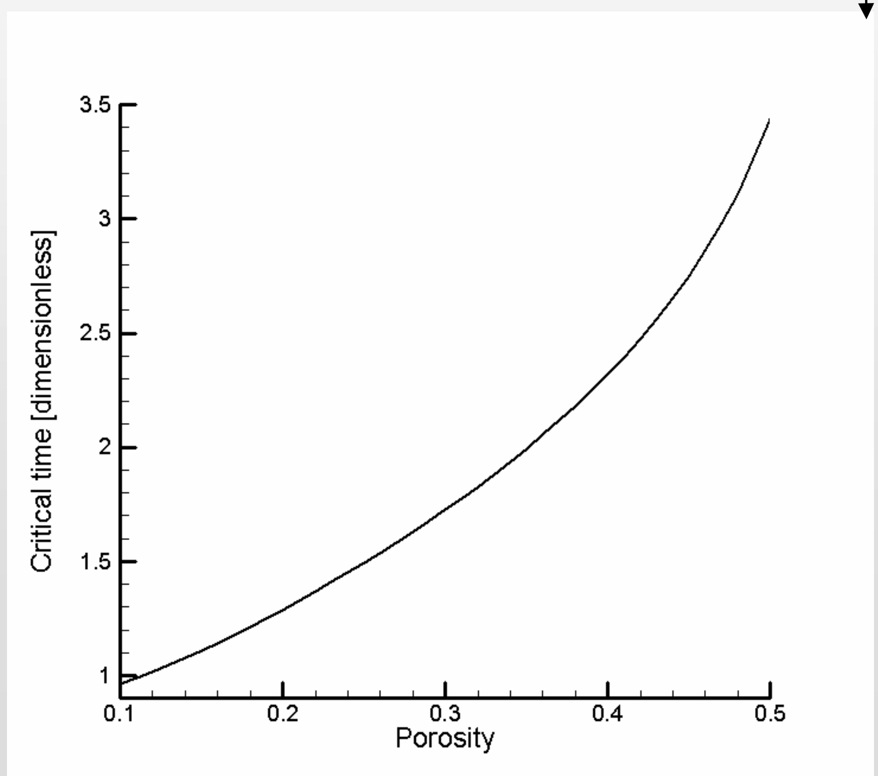
$$\nu = 0.25, \quad K_d = \frac{K_s}{1+50n}.$$



Critical time t‘ as the function of porosity for the initial amplitude

$$A_0 = \frac{\sqrt{2}}{2-\frac{1}{2}l^{S}{}'(n=0.55)}.$$

Critical time t‘ as the function of initial amplitude for *n=0.25*



Concluding remarks

Dependence of material parameters on porosity yields the existence of the threshold if the initial ampitude produces tension.

The size of the critical time and the intensity of tensile amplitude depends on the slope of the diagram of material parameters in function of porosity.