Weierstrass Institute for Applied Analysis and Stochastics
in Forschungsverbund Berlin e.V., Mohrenstrasse 39, D - 10117 Berlin, Germany

The enormous and the minute are interchangable manifestations of the eternal

William Blake (1757 – 1827)

The Parable of the Wise and Foolish Virgins

„To see a World in a Grain of Sand And a Heaven in a Wild Flower, Hold Infinity in the palm of your hand And Eternity in an hour“

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Przejścia mikro-makro w modelowaniu ośrodków granulowanych

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Wykład w Zakładzie Teorii Ośrodków Ciągłych
Instytut Podstawowych Problemów Techniki PAN

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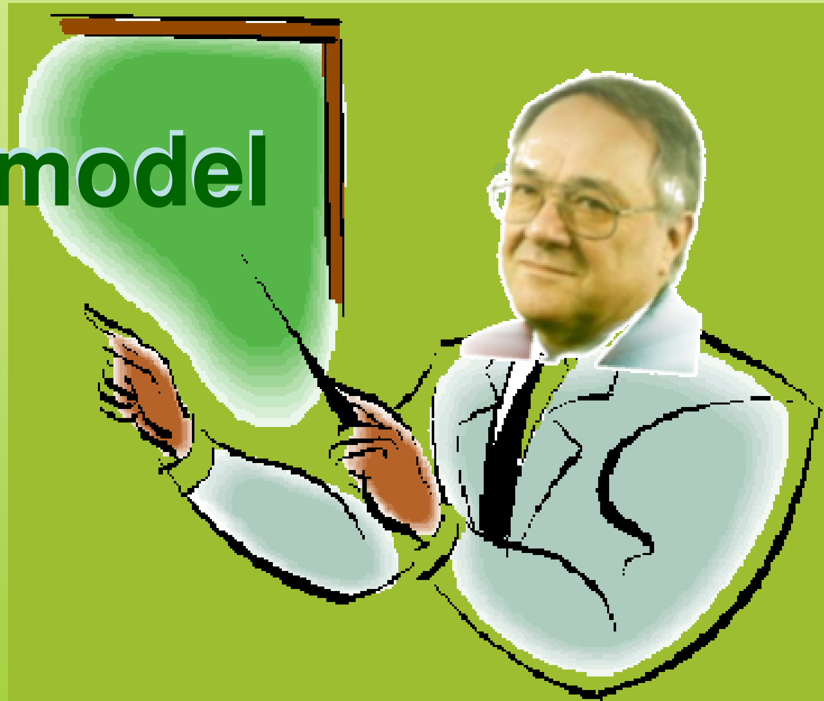
1. Linear poroelastic model



Macroscopic unknown fields of the linear model

$$\left\{\rho^F, \mathbf{v}^S, \mathbf{v}^F, \mathbf{e}^S, n\right\}$$

- partial mass density of the fluid
- velocity of the skeleton
- velocity of the fluid
- Almansi-Hamel deformation tensor of the skeleton
- porosity

Balance equations – linear model

$$\frac{\partial \rho^S}{\partial t} + \rho_0^S \, div \, \mathbf{v}^S = 0, \quad \frac{\partial \rho^F}{\partial t} + \rho_0^F \, div \, \mathbf{v}^F = 0,$$

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = div \, \mathbf{T}^S + \pi(\mathbf{v}^F - \mathbf{v}^S) + \rho^S \mathbf{b}^S,$$

$$\rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = div \, \mathbf{T}^F - \pi(\mathbf{v}^F - \mathbf{v}^S) + \rho^F \mathbf{b}^F,$$

$$\frac{\partial (n - n_E)}{\partial t} + \Phi \, div \, (\mathbf{v}^F - \mathbf{v}^S) = -\frac{n - n_E}{\tau}.$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = sym \, grad \, \mathbf{v}^S.$$

Linear constitutive relations

Partial stresses

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S e\mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon\mathbf{1} - \mathrm{N}(n - n_0)\mathbf{1},$$

$$\mathbf{T}^F = -p_{\mathrm{int}}^F \mathbf{1} + \mathrm{N}(n - n_0)\mathbf{1}, \quad p_{\mathrm{int}}^F = p_0^F - (\rho_0^F \kappa\varepsilon + Qe),$$

$$e := tr\mathbf{e}^S, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F}.$$

Porosity

$$n_E = n_0(1 + \delta e), \quad \Phi = const.$$

material constants

Linear field equations

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \lambda^S grad\, e + 2\mu^S div\, \mathbf{e}^S + Q grad \varepsilon + \pi(\mathbf{v}^F - \mathbf{v}^S) - \mathrm{N} grad\, n,$$

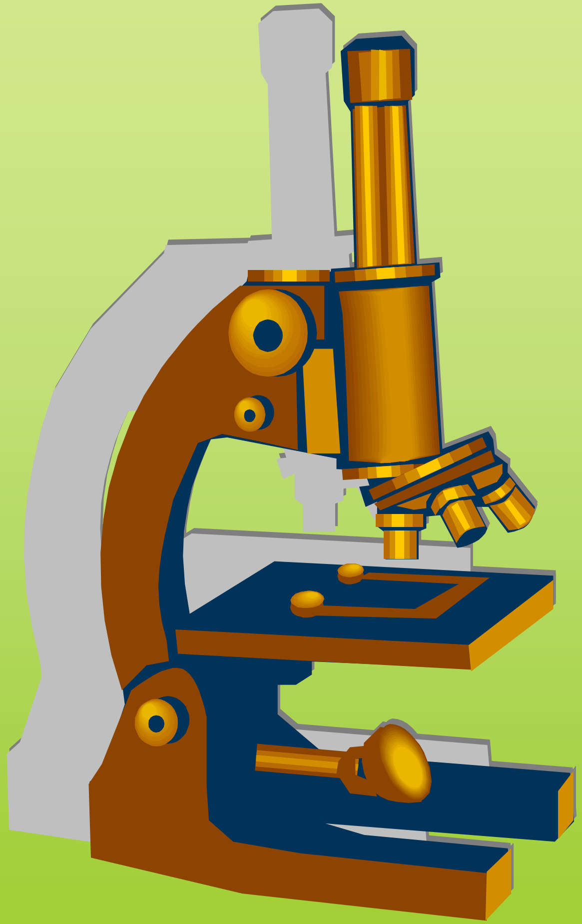
$$\rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = Q grad\, e + \rho_0^F \kappa\, grad \varepsilon - \pi(\mathbf{v}^F - \mathbf{v}^S) + \mathrm{N} grad\, n,$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = sym\, grad\, \mathbf{v}^S, \quad \frac{\partial \varepsilon}{\partial t} = div\, \mathbf{v}^F, \quad \frac{\partial}{\partial t}\left[n - n_0\delta e + \Phi(e - \varepsilon)\right] = -\frac{n - n_0 - n_0\delta e}{\tau}.$$

1) How to find material constants in „simple“ laboratory and field experiments?

2) How to find microstructural properties such as porosity, permeability or saturation?

Micro-macrotransitions?
Statistical averaging (REV)?
Time averaging?
Kinetic theory and macroscopic moments?



2. Micro-macrotransitions

(homogeneous microstructure)

Geometrical compatibility

Micro-macrorelations for partial mass densities in homogeneous microstructure

e.g.:

$$\rho^F = \frac{1}{V}\int_{REV(\mathbf{x})} \rho^{FR}(\mathbf{z},t) H(\mathbf{z},t)\, dV_{\mathbf{z}} \equiv n(\mathbf{x},t)\rho^{FR}(\mathbf{x},t),$$

$$n(\mathbf{x},t) := \frac{1}{V}\int_{REV(\mathbf{x})} H(\mathbf{z},t)\, dV_{\mathbf{z}}, \quad V := \text{volume}\, REV$$

homogeneity

where $H(\mathbf{z},t)$ is the characteristic function for the fluid component. Then

$$\rho^F = n\rho^{FR}, \quad \rho^F = \rho_0^F (1+\varepsilon)^{-1}, \quad \rho^{FR} = \rho_0^{FR}\left(1+\varepsilon^R\right)^{-1} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{n}{n_0} = \frac{1+\varepsilon^R}{1+\varepsilon}, \quad n_0 = \frac{\rho_0^F}{\rho_0^{FR}},$$

$$\rho^S = (1-n)\rho^{SR}, \quad \rho^S = \rho_0^S (1+e)^{-1}, \quad \rho^{SR} = \rho_0^{SR}\left(1+e^R\right)^{-1} \quad \Rightarrow$$

$$\Rightarrow \quad \frac{1-n}{1-n_0} = \frac{1+e^R}{1+e}.$$

Linearity (small deformations) yields **geometrical compatibility** conditions:

$$e = e^R + \frac{n-n_0}{1-n_0}, \quad \varepsilon = \varepsilon^R - \frac{n-n_0}{n_0}. \quad (1)$$

Changes of porosity

Constitutive relations:

- macro

$$
\begin{aligned}
&p^S - p_0^S = -(\lambda^S + \tfrac{2}{3}\mu^S)e - Q\varepsilon + \mathrm{N}(n\text{-}n_0),\\
&p^F - p_0^F = -Qe - \rho_0^F \kappa\varepsilon - \mathrm{N}(n\text{-}n_0),\\
&\text{equilibrium:}\quad \Delta p = (p^S - p_0^S) + (p^F - p_0^F) =\\
&\qquad = -(\lambda^S + \tfrac{2}{3}\mu^S + Q)e - (\rho_0^F \kappa + Q)\varepsilon.
\end{aligned}
\qquad (2)
$$

- micro

$$
(3)\qquad
\begin{aligned}
&p^{FR} - p_0^{FR} = -K_f \varepsilon^R, \quad p^{SR} - p_0^{SR} = -K_s e^R,\\
&\text{equilibrium:}\quad \Delta p = n_0(p^{FR} - p_0^{FR}) + (1 - n_0)(p^{SR} - p_0^{SR}).
\end{aligned}
$$

It follows

$$
\boxed{\frac{n - n_0}{n_0} = \delta e + \gamma(e - \varepsilon)}, \quad \delta := \frac{K_V - K}{n_0(K_s - K_f)}, \quad \gamma := \frac{\rho_0^F \kappa + Q - n_0 K_f}{n_0(K_s - K_f)},
$$

$$
K_V := (1 - n_0)K_s + n_0 K_f, \quad K := \lambda^S + \tfrac{2}{3}\mu^S + \rho_0^F \kappa + 2Q.
$$

Changes of porosity
- the solution of the porosity equation

$$\frac{\partial(n-n_E)}{\partial t}+\frac{n-n_E}{\tau}=-\Phi\, div(\mathbf{v}^F-\mathbf{v}^S), \quad n_E=n_0(1+\delta\, e).$$

Mass balance equations:

$$div\mathbf{v}^S=-\frac{\partial}{\partial t}\frac{\rho^S}{\rho_0^S}=\frac{\partial e}{\partial t}, \quad div\mathbf{v}^F=-\frac{\partial}{\partial t}\frac{\rho^F}{\rho_0^F}=\frac{\partial \varepsilon}{\partial t}.$$

Hence

$$\frac{n-n_0}{n_0}=\delta e+\frac{\Phi}{n_0}(e-\varepsilon)-\frac{\Phi}{n_0\tau}\int_0^t(e-\varepsilon)(s)e^{-\frac{t-s}{\tau}}ds.$$

memory effect

The micro-macrorelation follows provided

$$\tau\to\infty, \quad \gamma=\frac{\Phi}{n_0}.$$

Gedankenexperiments
for homogeneous microstructures
Preliminaries – Biot‘s notation

Change of the fluid content:

Fluid contained in a reference volume dV_0 of the skeleton

- reference configuration: $\rho_0^F dV_0$
- current configuration: $\rho^F (1+e) dV_0$
- normalized change – **increment of the fluid content**:

$$\frac{1}{\rho_0^{FR}}\left[(1+e)\rho^F - \rho_0^F\right]dV_0 := \zeta dV_0 = n_0\left[\frac{1+e}{1+\varepsilon} - 1\right]dV_0 \quad \Rightarrow \quad \boxed{\zeta \approx n_0(e-\varepsilon).}$$

Bulk stress and pore pressure

$$\mathbf{T} \approx \mathbf{T}^S + \mathbf{T}^F = \mathbf{T}_0 + (H - 2\mu^S)e\mathbf{1} + 2\mu^S \mathbf{e}^S - C\zeta\mathbf{1},$$

$$p := -\tfrac{1}{3} tr\mathbf{T} = p_0 - Ke + C\zeta,$$

$$p_f := \frac{p^F}{n_0} = p_f^0 - Ce + M\zeta - \mathrm{N}\frac{n-n_0}{n_0}.$$

Biot‘s constants

$$H := \lambda^S + 2\mu^S + \rho_0^F \kappa + 2Q =$$

$$= K + \tfrac{4}{3}\mu^S,$$

$$C := \tfrac{1}{n_0}(Q + \rho_0^F \kappa), \quad M := \frac{\rho_0^F \kappa}{n_0^2}.$$

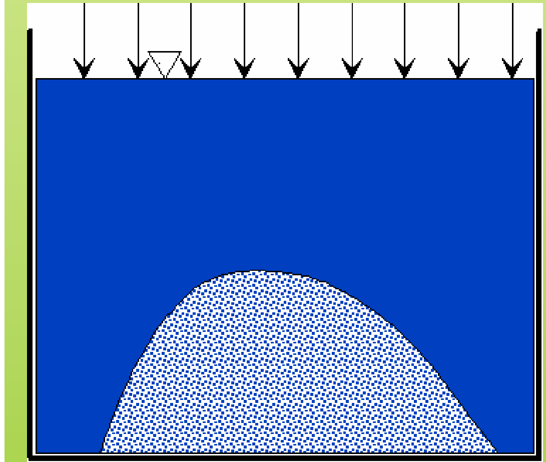
Gedankenexperiments for homogeneous microstructures

Unknown: $\left\{ e, \zeta, n, e^R, \varepsilon^R, p - p_0, p_f - p_f^0 \right\}$ **= 7;**

Equations: 2 geom., 1 equilib., 4 constit., 1 test = 8.

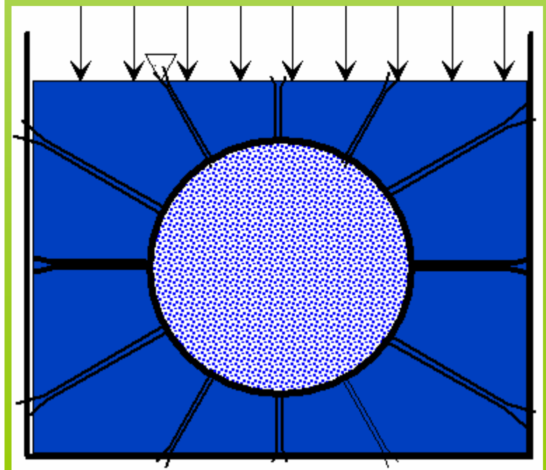
General equilibrium conditions:

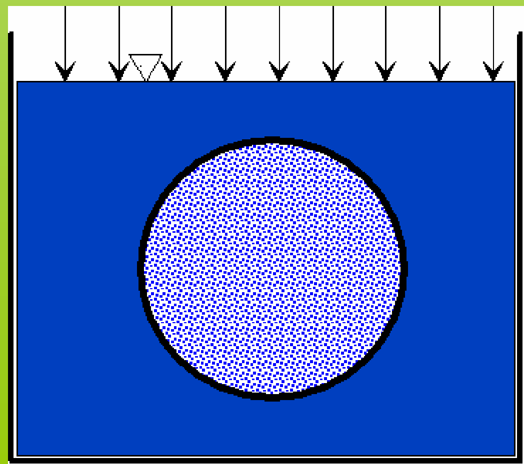
$$\Delta p = (p^S - p_0^S) + (p^F - p_0^F) =$$
$$= (1 - n_0)(p^{SR} - p_0^{SR}) + n_0 (p^{FR} - p_0^{FR}).$$



Unjacketed test:

$$p_f - p_f^0 = \Delta p,$$





Jacketed drained

$$p_f - p_f^0 = 0,$$

and undrained tests:

$$\zeta = 0, \quad \text{i.e.} \quad e = \varepsilon.$$

Solutions of field equations and geometrical compatibility

jacketed undrained

$$e = -\frac{\Delta p}{K},\quad \zeta = 0,\quad \frac{n-n_0}{n_0} = -\frac{C-K_f}{K(K_f-\mathrm{N})}\Delta p,$$

$$p_f - p_f^0 = \left(\frac{C}{K} + \mathrm{N}\frac{C-K_f}{K(K_f-\mathrm{N})}\right)\Delta p,$$

$$\boxed{K = K_V - n_0(K_s - K_f)\frac{C-K_f}{K_f-\mathrm{N}},}$$

(1)

$$K_V := (1-n_0)K_s + n_0 K_f.$$

$$C > K_f \quad \Rightarrow \quad \mathrm{N} < K_f.$$

jacketed drained

$$e = -\frac{\Delta p}{K_b} - \frac{\mathrm{N}C}{K_\mathrm{b}M}\frac{\Delta p}{K_n},$$

$$\zeta = -\frac{C}{K_b M}\left(1 + \frac{K\mathrm{N}}{CK_n}\right)\Delta p,$$

$$\frac{n-n_0}{n_0} = -K_n\Delta p,$$

$$K_b := K - \frac{C^2}{M},$$

$$K_n := K_s\frac{(1-n_0)\frac{\mathrm{N}C}{K_\mathrm{b}M} - n_0}{1-(1-n_0)\frac{K_s}{K_b}},$$

$$\boxed{\frac{K_s}{K_b}\left(n_0 - \frac{C}{M}\right) - \frac{K_s}{K_n}\left(n_0 - \frac{\mathrm{N}(K+C)}{K_b M}\right) = 0,}$$

(2) &

unjacketed

$$e = -\frac{1-\frac{C}{K_W}}{1-\frac{C}{K}}\frac{\Delta p}{K},\quad \zeta = -\frac{1-\frac{K}{K_W}}{1-\frac{C}{K}}\frac{\Delta p}{K},$$

$$\frac{n-n_0}{1-n_0} = \left(\frac{K}{K_s} - -\frac{1-\frac{C}{K_W}}{1-\frac{C}{K}}\right)\frac{\Delta p}{K},$$

(3)

$$\boxed{K = \frac{C-M+\frac{MK_b}{K_W}}{1-\frac{C}{K}} - \mathrm{N}\frac{1-\mathrm{n}_0}{\mathrm{n}_0}\left(\frac{K}{K_s} - \frac{1-\frac{C}{K_W}}{1-\frac{C}{K}}\right);}$$

$$\frac{1}{K_W} := \frac{1-n_0}{K_s} + \frac{n_0}{K_f}.$$

$$\boxed{K_d = K_b\left(1 - \frac{\mathrm{N}C}{K_n M}\right)^{-1}}$$

- given experimentally.

(4)

Full set of equations for *K, C, M, N*:

$$K = K_V - n_0(K_s - K_f)\frac{C-K_f}{K_f-\mathrm{N}}, \quad K_V := (1-n_0)K_s + n_0 K_f,$$

$$\frac{K_s}{K_b}\left(n_0 - \frac{C}{M}\right) - \frac{K_s}{K_n}\left(n_0 - \frac{\mathrm{N}(K+C)}{K_b M}\right) = 0, \quad K_b := K - \frac{C^2}{M},$$

$$K = \frac{C - M + \frac{MK_b}{K_W}}{1-\frac{C}{K}} - \mathrm{N}\frac{1\text{-}n_0}{n_0}\left(\frac{K}{K_s} - \frac{1-\frac{C}{K_W}}{1-\frac{C}{K}}\right), \quad \frac{1}{K_W} := \frac{1-n_0}{K_s} + \frac{n_0}{K_f},$$

$$K_d = K_b\left\{1 + \frac{\mathrm{N}C}{K_n M}\right\}^{-1}, \quad K_n := K_s \frac{(1-n_0)\frac{\mathrm{N}C}{K_b M} - n_0}{1-(1-n_0)\frac{K_s}{K_b}}.$$

A): For N=0:

$$\xi := \frac{K_s}{K_f} - 1.$$

B): N belongs to the set of macroscopic material parameters but it is small in comparison with other compressibilities - iteration

C) Full solution for *K, C, M,* N

$$K = \frac{(K_s - K_b)^2}{K_s(1+n_0\xi) - K_b} + K_b, \quad C = \frac{K_s(K_s - K_b)}{K_s(1+n_0\xi) - K_b},$$

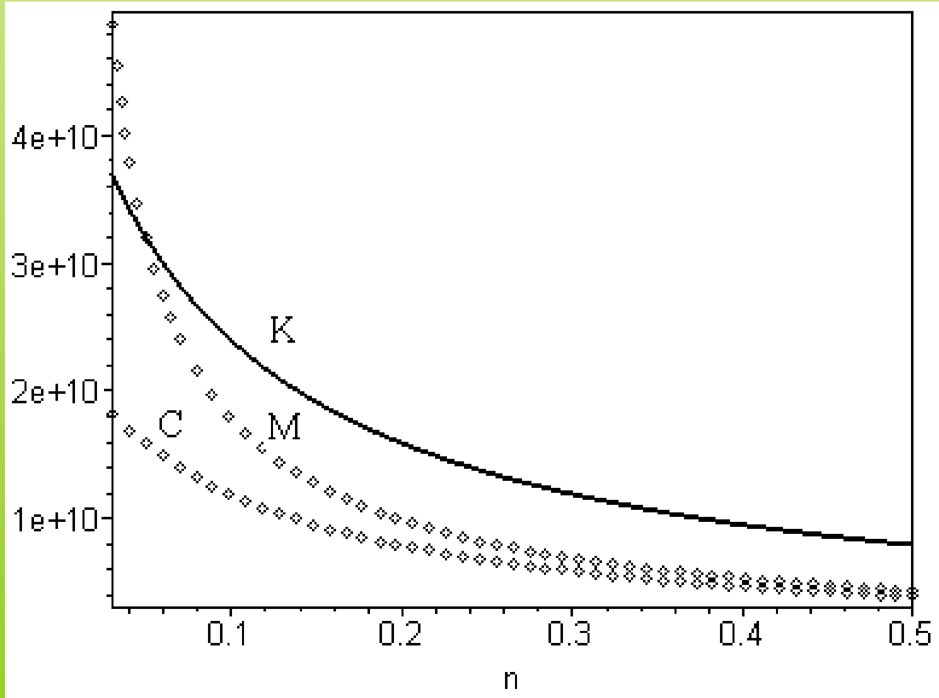
$$M = \frac{K_s^2}{K_s(1+n_0\xi) - K_b},$$

Numerical example

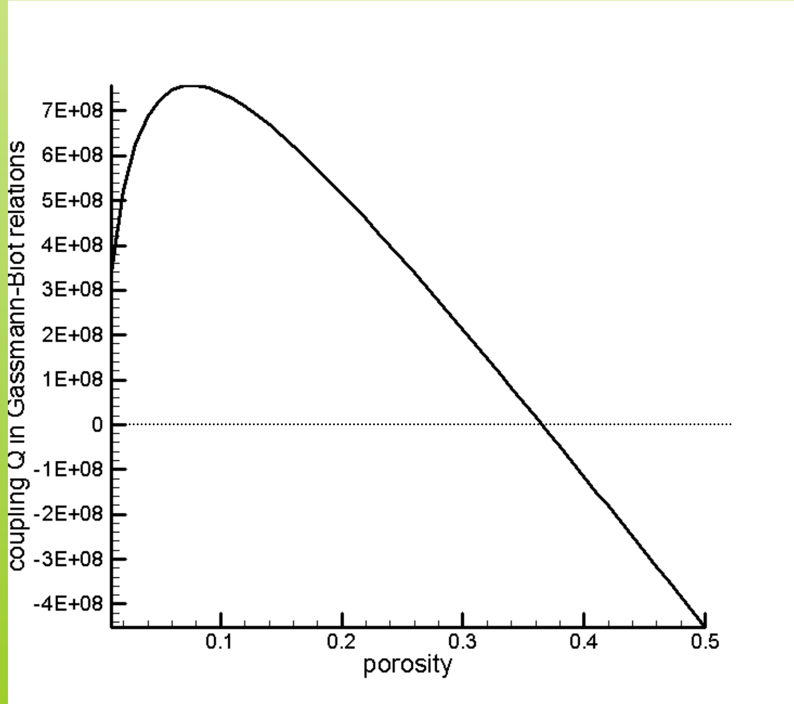
$$K_s = 48 \times 10^9 \, Pa, \quad K_f = 2.25 \times 10^9 \, Pa$$

$$\text{Geertsma (empirical)}: \quad K_b \text{ or } K_d = \frac{K_s}{1 + 20 n_0}.$$

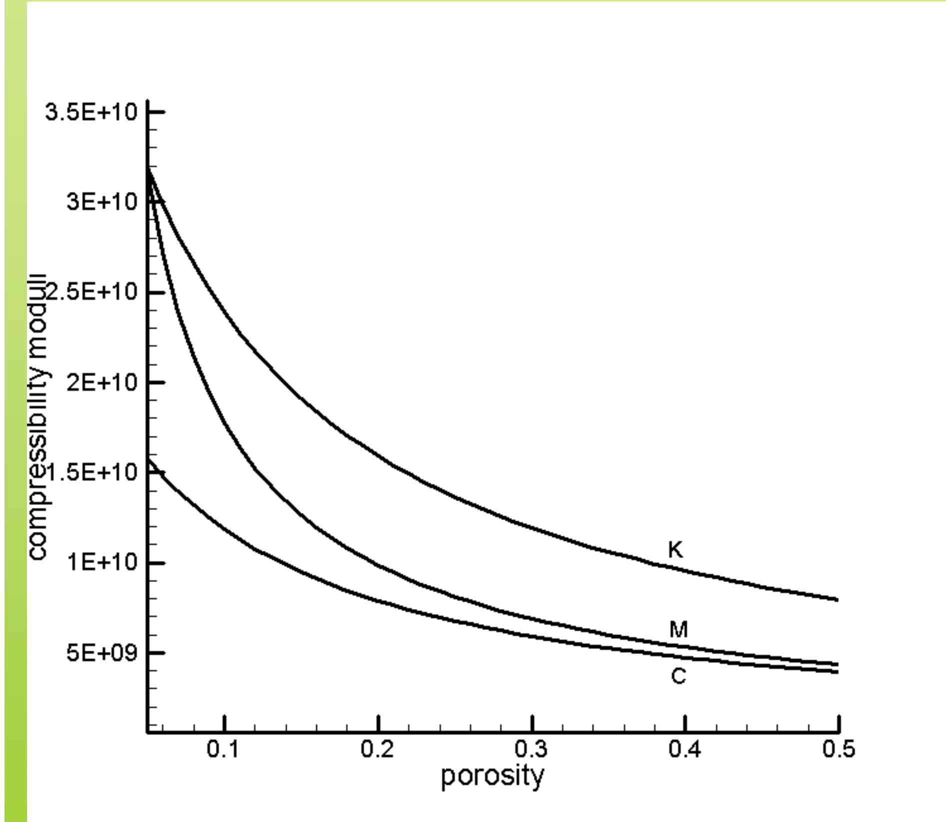
$$Q = n_0 (C - n_0 M).$$



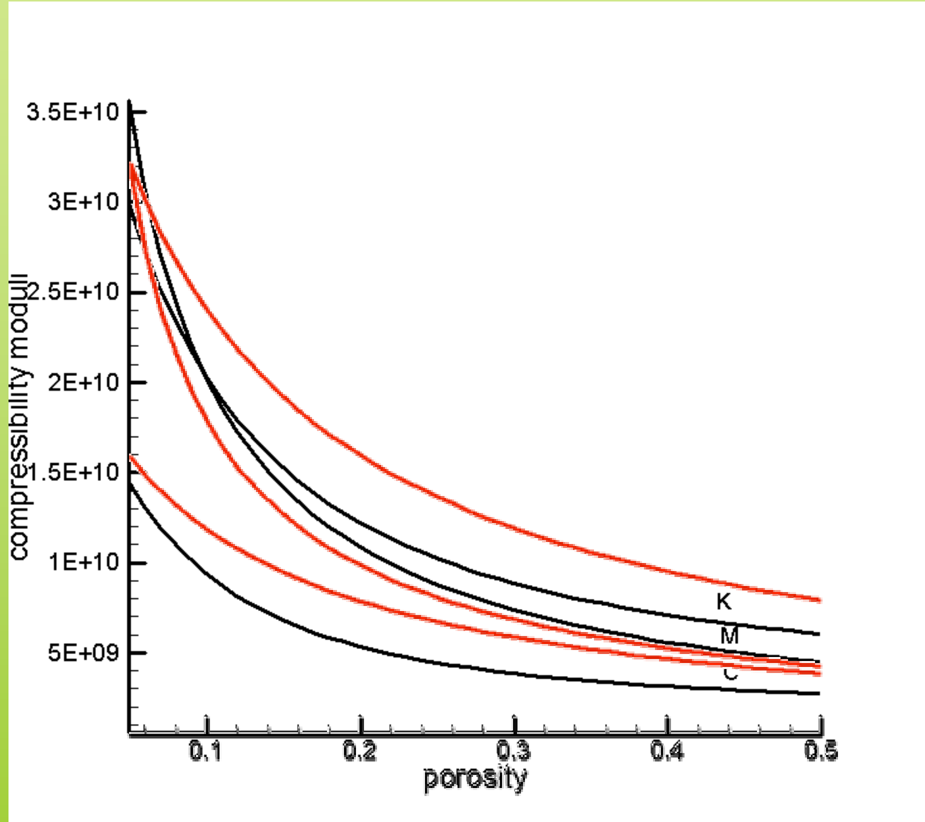
Material parameters *K, C, M* in the zeroth approximation (Gassmann)



Coupling parameter *Q* in Biots model



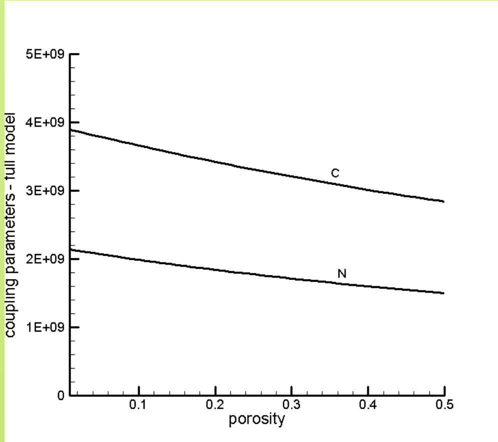
Material parameters *K, C, M* in the zeroth approximation (Gassmann)



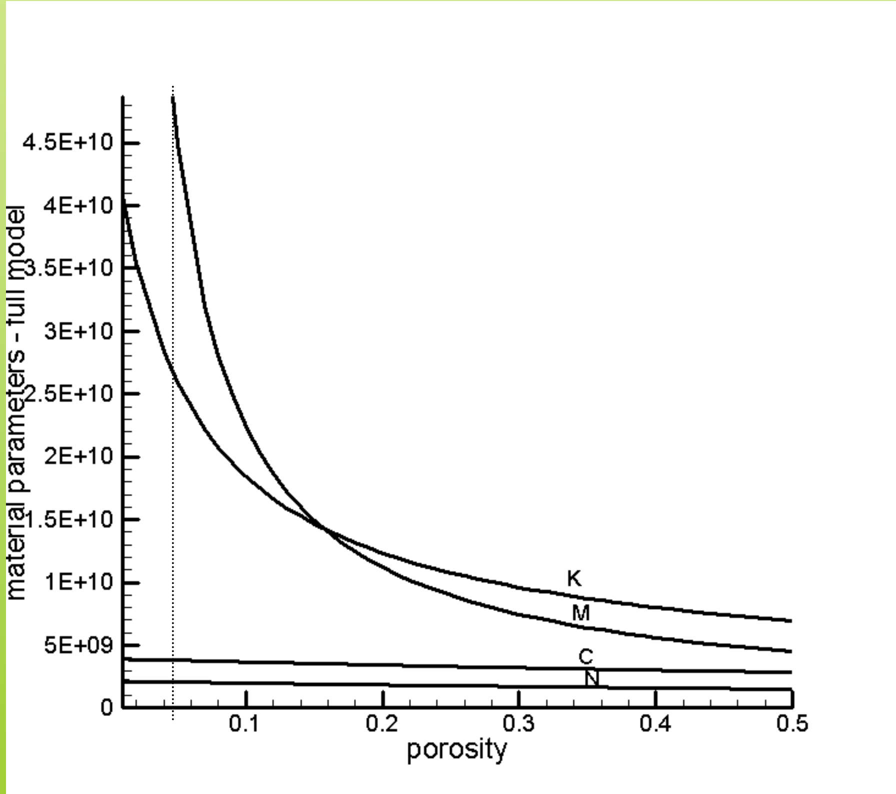
Material parameters *K, C, M* in the first approximation (Gassmann)

Full model:

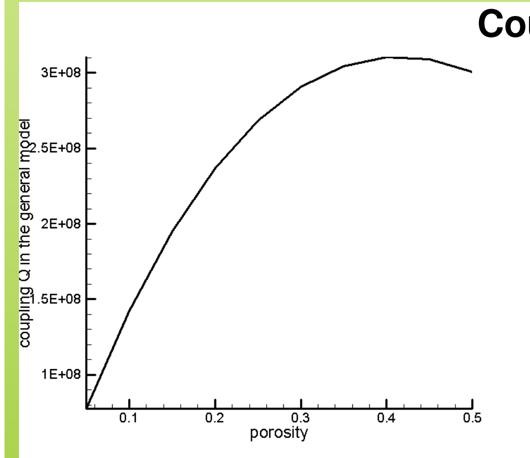
$$\text{Geertsma}: \quad K_d = \frac{K_s}{1+20n}$$



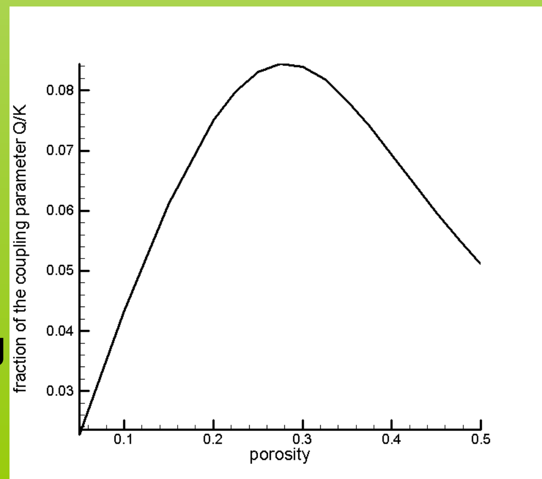
Coupling parameters *C, N* in the full model



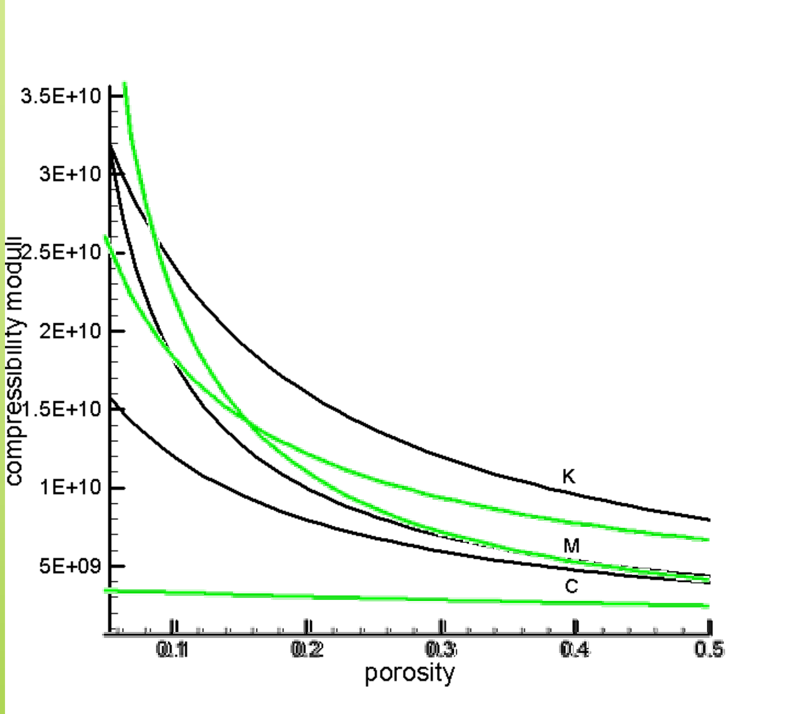
Material parameters *K, C, M, N* in the full model



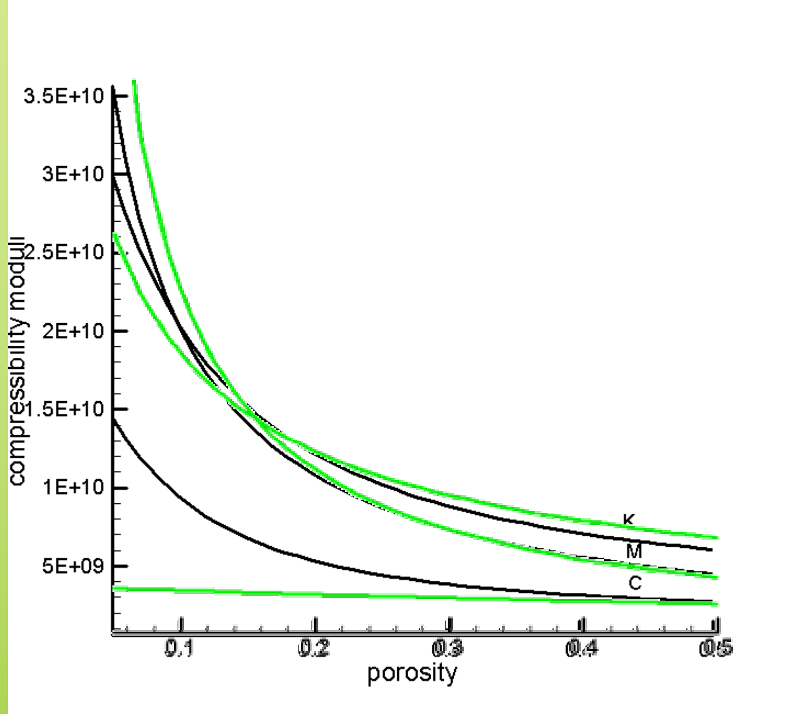
Coupling parameter *Q* in the full model



Fraction of coupling parameter: *Q/K* in the full model



Material parameters *K, C, M* in the zeroth approximation (Gassmann)



Material parameters *K, C, M* in the first approximation (Gassmann)

Comparison with the full solution (green)

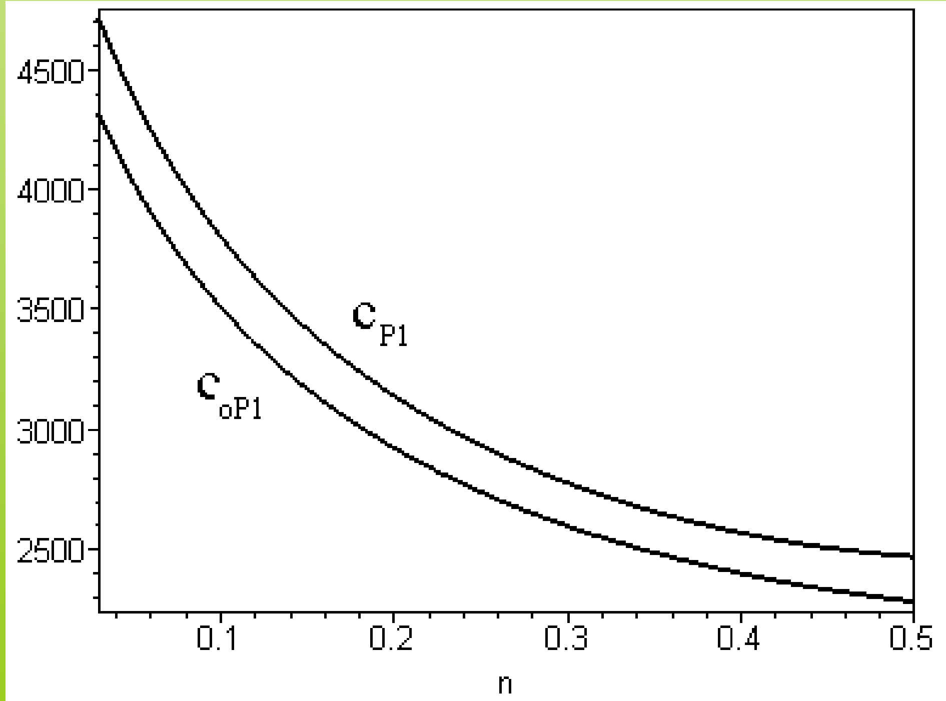
Carlo Lai (July 2002)

$$\rho_0^S = (1 - n_0) \times 4000 \frac{kg}{m^3},$$

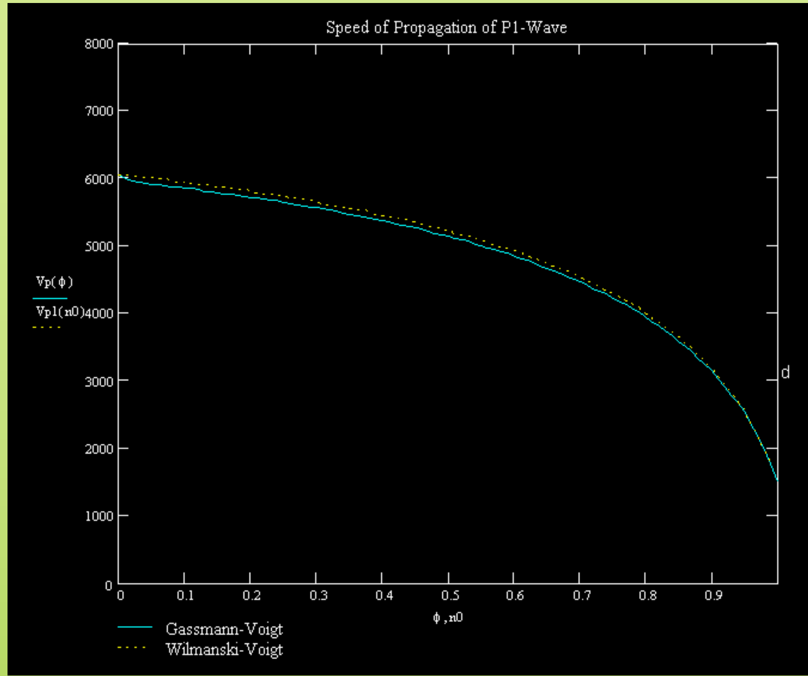
$$\rho_0^F = n_0 \times 1000 \frac{kg}{m^3},$$

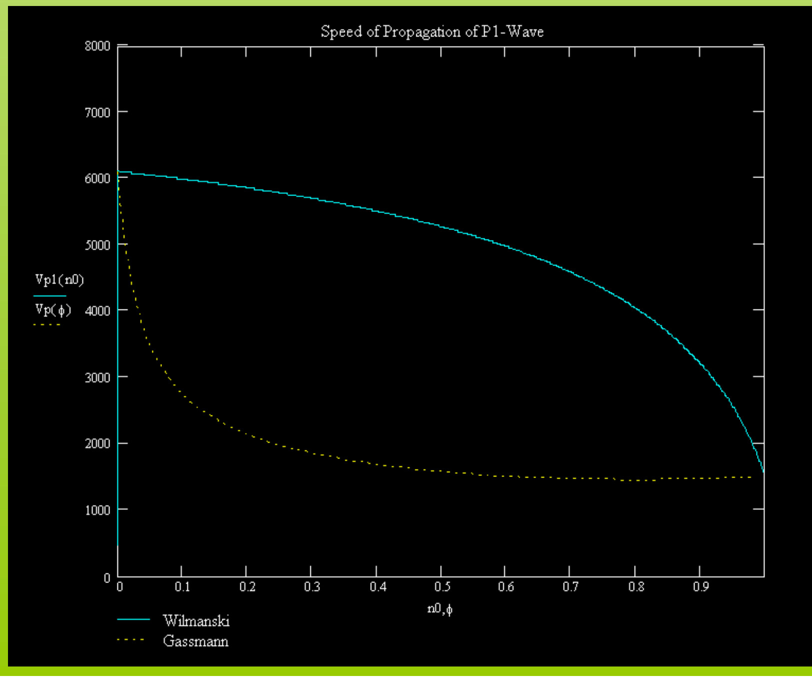
$$\text{Poisson number}: \quad \nu = 0.2.$$

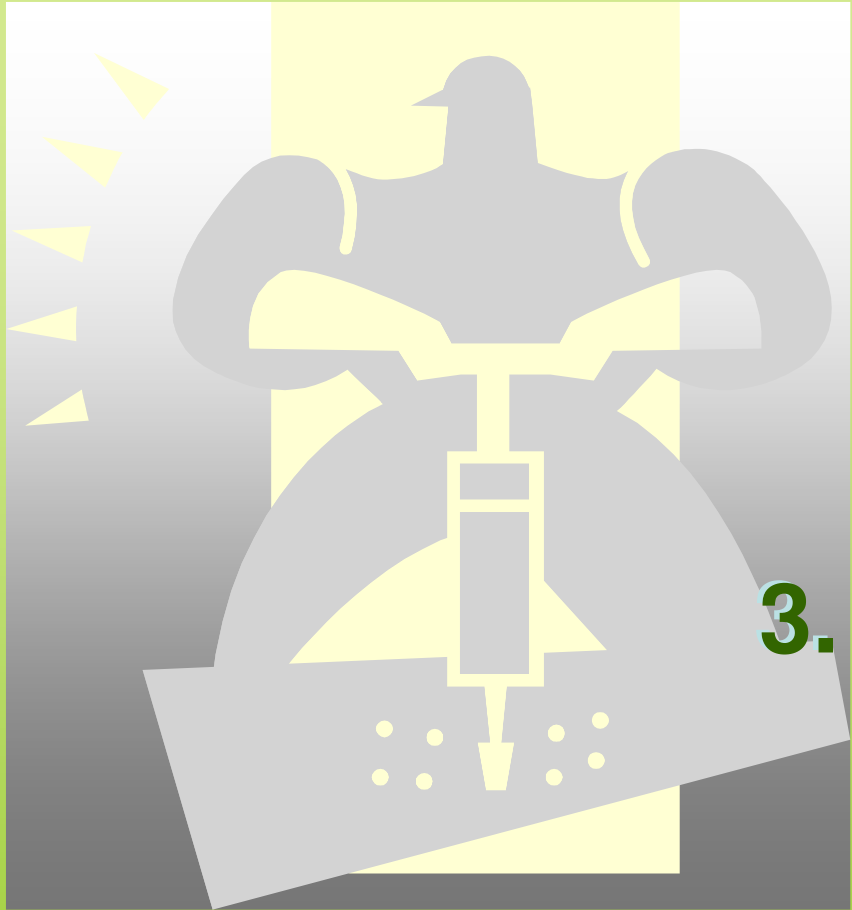
Some results for P1-waves:



speed of propagation of P1-wave
in the low frequency approximation







3. Dependence of μ^S, K_d and K_b on porosity

In the drained simple test the material behaves as there were no fluid in pores. Hence the stress – strain relation should be the same as in the classical elasticity with the compressibility modulus K_d and shear modulus μ^S, i.e.

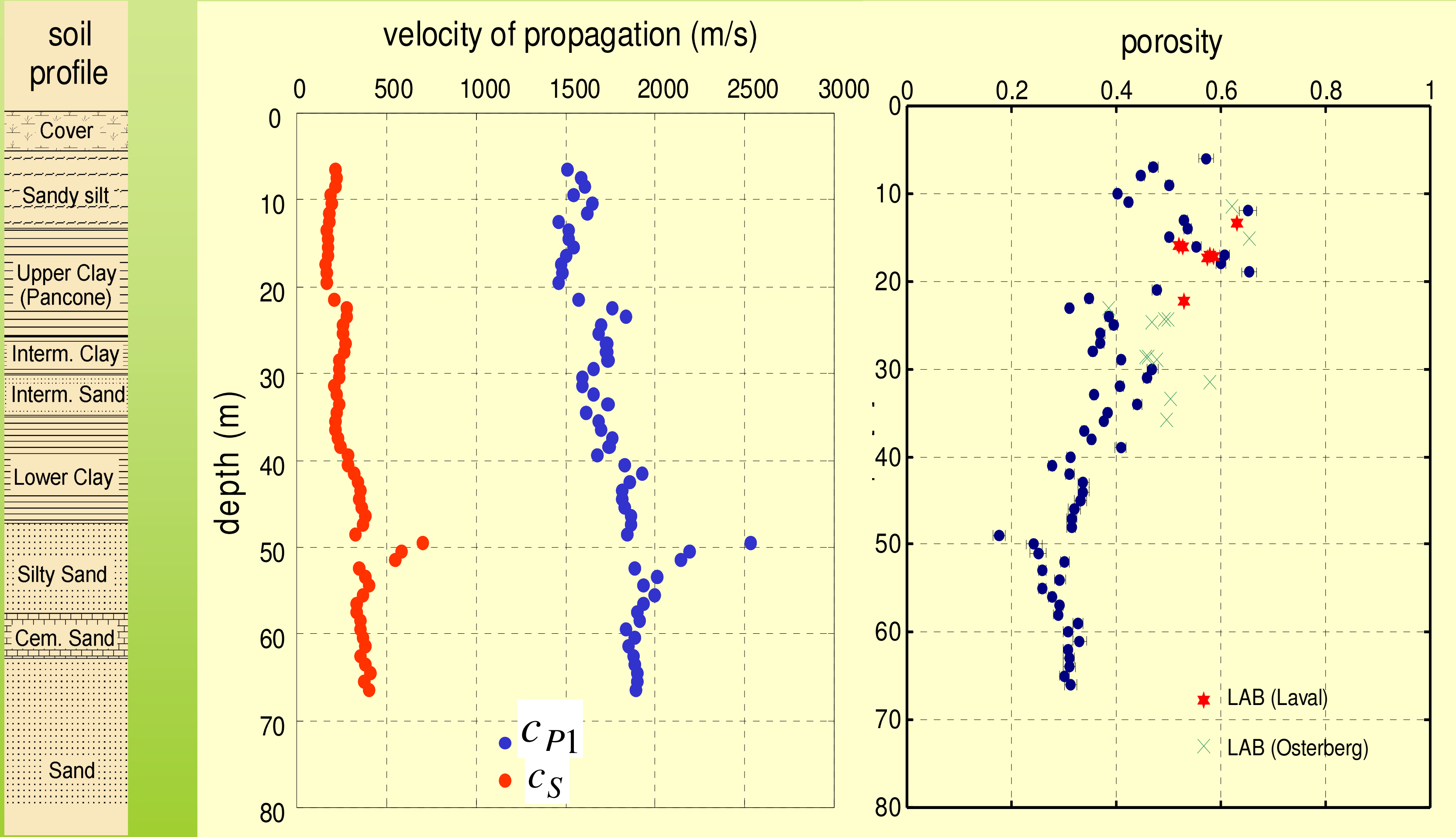
$$\mu^S = 3K_d \frac{1-2\nu}{2(1+\nu)} = 3K_b \left.\frac{1-2\nu}{2(1+\nu)}\right|_{N=0},$$

where ν is the Poisson's number.

Simultaneously speeds of propagation react weakly on changes of Poisson's number – see: results of Carlo Lai. Hence we can choose, for instance, $\nu=0.2$ and either calculate K_d (K_b) from the above formula provided μ^S was measured by the speed of shear wave or calculate μ^S provided K_d (K_b) is given empirically.

<u>Important practical case</u>: given two speeds of propagation (P1- and shear wave) and the Poisson's number. Then by means of MICRO-MACRO one can calculate porosity (inverse problem).

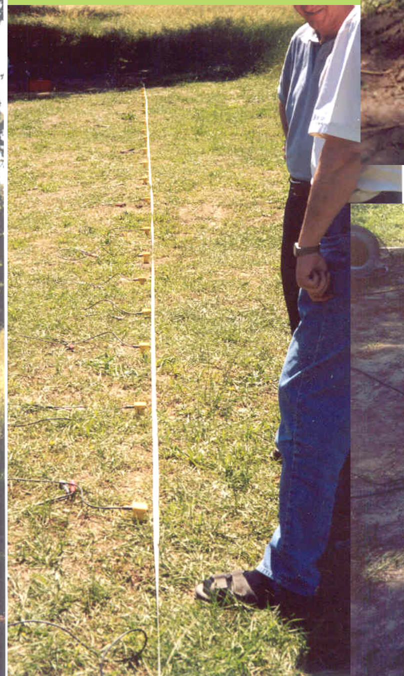
Experimental measurements: c_{P1} ,c_S and porosity profiles



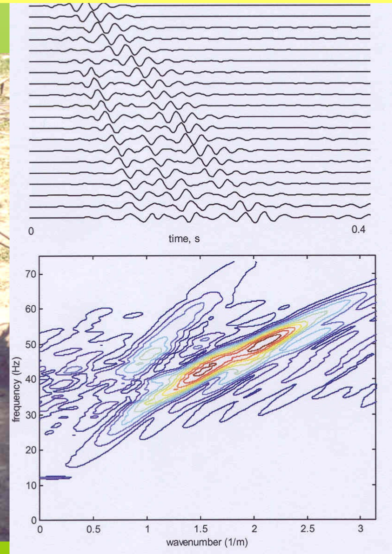
Pisa site. After: FOTI,S., LAI, C., LANCELLOTTA, R.; Porosity of Fluid-saturated Porous Media from Measured Seismic Wave Velocities, Géotechnique, Vol. 52, No. 5, 359-373 (2002).

LAI, C.; Recent Advances in the Solution of Some Parameter-Identification Problems Relevant To Soil Dynamics, Lecture, Oct. 21, 2002, ROSE School, Pavia, Italy.

Testing of soils by surface waves (SASW)



Politecnico di Torino, 2003



4. Conclusions

1. Biot's model follows as a limit case of a general thermodynamic model provided the relaxation of porosity is neglected.

2. Gassmann – like relations, derived for the extended model with a dependence on porosity gradient by means of the Micro-MacroTransition for homogeneous microstructure satisfy ALL compatibility relations for simple tests and yield in the limit the classical Gassmann relations.

3. In order to fulfil geometrical compatibility relations of micro-macrotransition the porosity balance equation must be corrected on an equilibrium contribution to the source term.

4. Micro-MacroTransition gives rise to a possibility of a natural description of processes in unsaturated granular materials.

Newton (1795)

The Ghost of a Flea (ca. 1819)