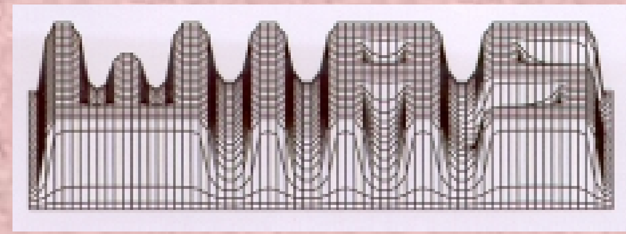

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Thermodynamic modeling of porous materials

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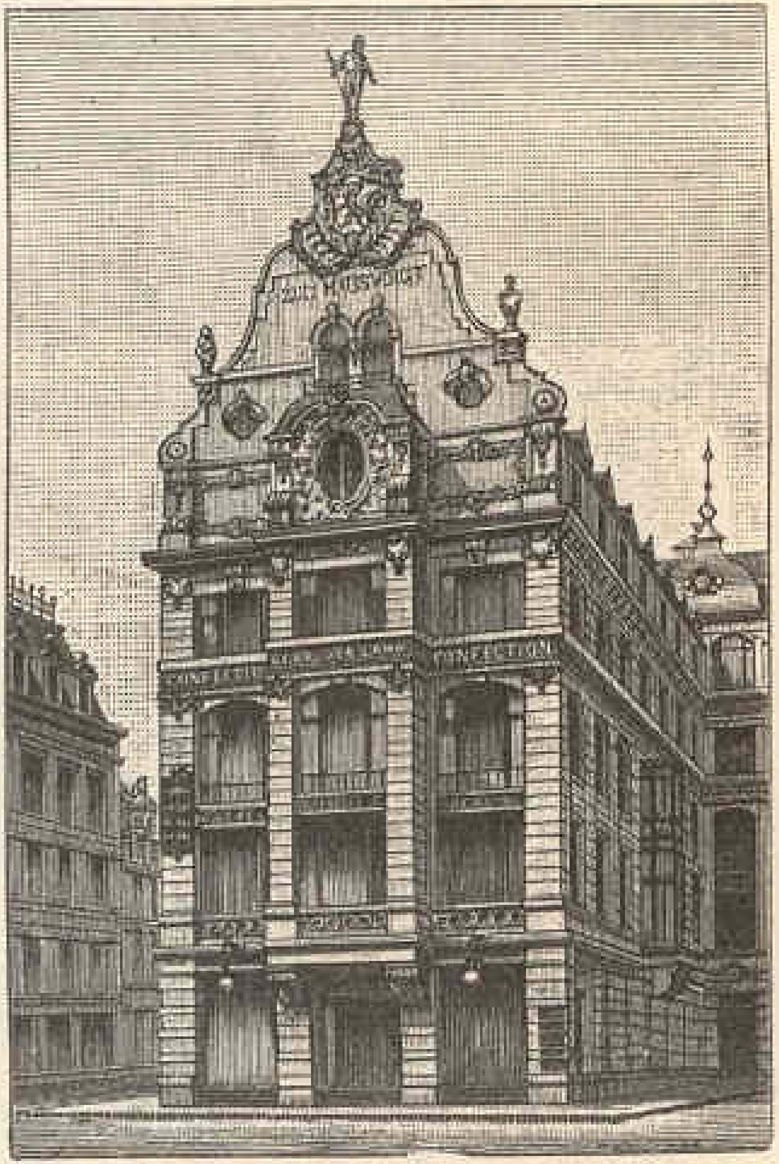

4. Kaufhaus ‚Zum Hausvoigt‘ in Berlin.

app. 1907

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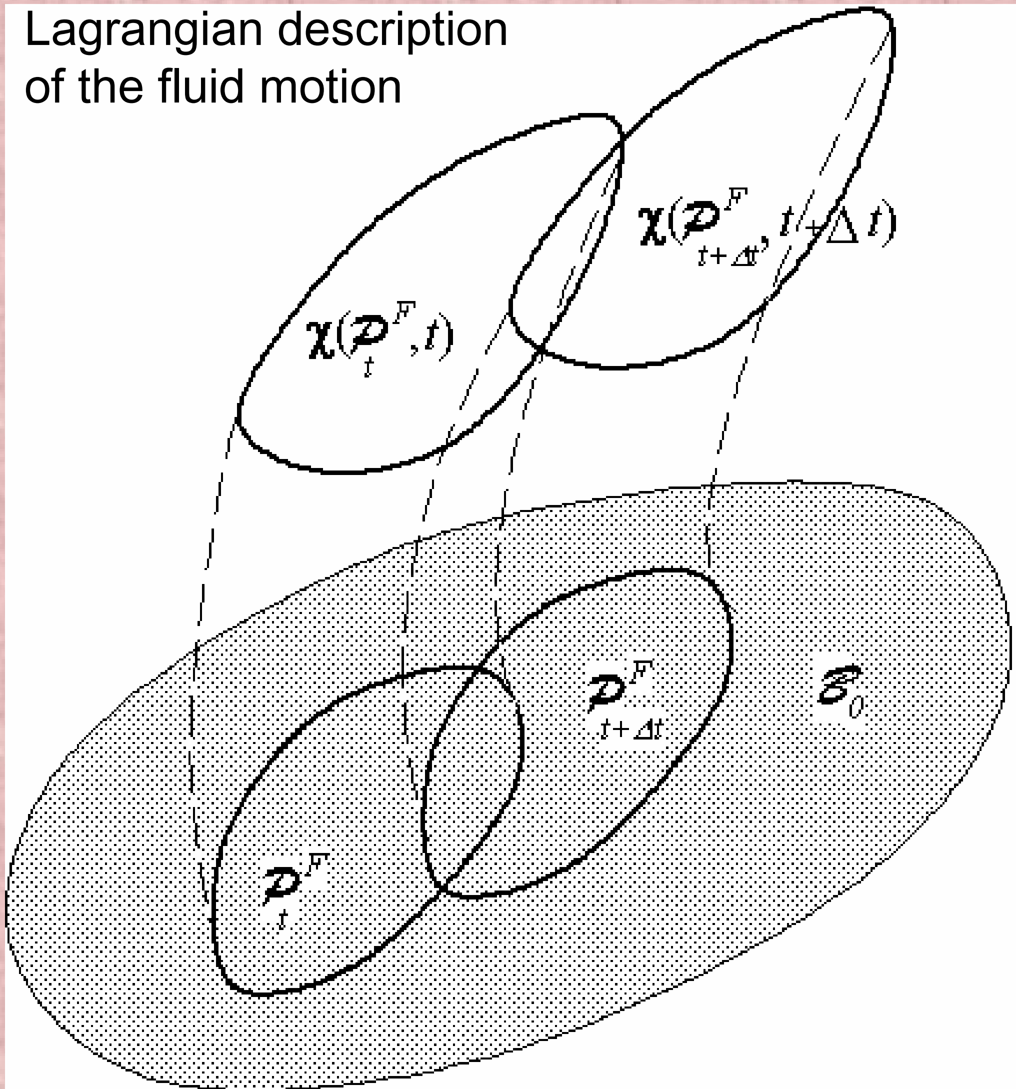
Kinematics*



Reference configuration of the skeleton and projections of material elements of the fluid component

Motion of the skeleton (Lagrange):

$$\mathbf{x} = \chi(\mathbf{X}, t) \quad \Rightarrow$$

$$\Rightarrow \quad \mathbf{x}'^S := \frac{\partial \chi}{\partial t}, \quad \mathbf{F}^S := Grad\, \chi.$$

Motion of the fluid component (Euler):

$$\mathbf{v}^F = \mathbf{v}^F(\mathbf{x}, t) \quad \Rightarrow$$

$$\Rightarrow \quad \mathbf{x}'^F := \mathbf{v}^F(\chi(\mathbf{X}, t), t).$$

Lagrangian relative velocity:

$$\mathbf{X}'^F := \mathbf{F}^{S-1}\left(\mathbf{x}'^F - \mathbf{x}'^S\right).$$

*Wilmanski, K.; *On weak discontinuity waves in porous materials,* in: M. Marques, J. Rodrigues (eds.), *Trends in Applications of Mathematics to Mechanics*, 71-83, Longman (1994),
Wilmanski, K.; *Lagrangean model of two-phase porous material*, J. Non-Equil. Thermodyn., **20**, 50-77 (1995),
Wilmanski, K.; *Thermomechanics of Continua*, Springer (1998).

Derivation:

motion of a fluid particle $\mathbf{x}' = \mathbf{x} + \mathbf{v}^F \Delta t \quad \Rightarrow$

Lagrange

$$\chi(\mathbf{X} + \Delta\mathbf{X}, t + \Delta t) = \chi(\mathbf{X}, t) + \mathbf{F}^S \Delta\mathbf{X} + \mathbf{x}'^S \Delta t$$

$$= \chi(\mathbf{X}, t) + \mathbf{x}'^F \Delta t.$$

Consequently

$$\mathbf{X}'^F := \lim_{\Delta t \to 0} \frac{\Delta \mathbf{X}}{\Delta t} = \mathbf{F}^{S-1}\left(\mathbf{x}'^F - \mathbf{x}'^S\right).$$

Time differentiation of material objects:

material
with respect to the skeleton

$$\frac{d}{dt} \int_{P_t^S} \varphi(\mathbf{X}, t)\, dt = \int_{P_t^S} \frac{\partial \varphi}{\partial t}(\mathbf{X}, t)\, dt,$$

material
with respect to the fluid

$$\frac{d}{dt} \int_{P_t^F} \varphi(\mathbf{X}, t)\, dt = \int_{P_t^F} \frac{\partial \varphi}{\partial t}(\mathbf{X}, t)\, dt + \oint_{\partial P_t^F} \varphi(\mathbf{X}, t)\, \mathbf{N} \cdot \mathbf{X}'^F\, dt.$$

Fields for a two-component poroelastic material

$$\left\{\rho^F, \rho^S, \mathbf{x}'^F, \mathbf{x}'^S, \mathbf{F}^S, n, \theta\right\} \in \mathrm{V}^{19},$$

$$\left(\mathbf{X}, t\right) \in \mathrm{B} \times \mathrm{T}, \quad \mathrm{B} \times \mathrm{T} \to \mathrm{V}^{19},$$

where

ρ^F - mass density of the fluid component in the reference configuration of the skeleton,

ρ^S - mass density of the skeleton in its reference configuration,

$\mathbf{x}'^F$ - velocity of the fluid component,

$\mathbf{x}'^S$ - velocity of the skeleton,

$\mathbf{F}^S$ - deformation gradient of the skeleton, $\mathbf{F}^S = \mathbf{1}$ in the reference configuration,

n - porosity (volume fraction of the fluid component),

θ - absolute temperature common for both components.

Partial balance equations*

$$\mathrm{B}^{\rho^S} = \frac{\partial \rho^S}{\partial t} - \hat{\rho} = 0, \quad \mathrm{B}^{\rho^F} = \frac{\partial \rho^F}{\partial t} + Div\left(\rho^F \mathbf{X}'^F\right) + \hat{\rho} = 0.$$

$$\mathrm{B}^{\mathbf{v}^S} = \frac{\partial\left(\rho^S \mathbf{x}'^S\right)}{\partial t} - Div\,\mathbf{P}^S - \hat{\mathbf{p}} = 0,$$

$$\mathrm{B}^{\mathbf{v}^F} = \frac{\partial\left(\rho^F \mathbf{x}'^F\right)}{\partial t} + Div\left(\rho^F \mathbf{x}'^F \otimes \mathbf{X}'^F - \mathbf{P}^F\right) + \hat{\mathbf{p}} = 0.$$

$$\mathrm{B}^{\mathbf{F}^S} = \frac{\partial \mathbf{F}^S}{\partial t} - Div\left(\mathbf{x}'^S \otimes \mathbf{1}\right) = 0,$$

with $Grad\,\mathbf{F}^S = \left(Grad\,\mathbf{F}^S\right)^{T}$ [23]

$$\Rightarrow \exists\, \mathbf{x} = \chi(\mathbf{X}, t) \text{ s.t. } \mathbf{F}^S \equiv Grad\,\chi,\ \mathbf{x}'^S \equiv \frac{\partial \chi}{\partial t},$$

$$\mathrm{B}^{n} = \frac{\partial n}{\partial t} + Div\,\mathbf{J} - \hat{n} = 0,$$

Alternative: *balance of equilibrated forces* – second order scalar differential equation introduced by Goodman & Cowin (1972)

$$\mathrm{B}^{\varepsilon} = \frac{\partial}{\partial t}\left(\rho\varepsilon + \tfrac{1}{2}\rho\,\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}\right) + Div\left[\rho^F\left(\varepsilon + \tfrac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}\right)\mathbf{X}'^F + \mathbf{Q} - \mathbf{P}^T\dot{\mathbf{x}}\right] = 0,$$

Wilmanski, K.; *Multicomponent models in geophysics*, in: J. Majewski, R. Teisseyre (eds.), *Earthquake Thermodynamics And Phase Transformations in the Earth's Interior*, 567-655, Academic Press (2001),
Wilmanski, K.; *Toward an extended thermodynamics of porous and granular materials*, in: G. Iooss, O. Gues, A. Nouri (eds.) *Trends in Applications of Mathematics to Mechanics*, Chapman&Hall, 147-160 (2000).

Definitions of mean quantities:

$$\rho := \rho^F + \rho^S, \quad \rho\,\dot{\mathbf{x}} := \rho^F \mathbf{x}'^F + \rho^S \mathbf{x}'^S,$$

$$\mathbf{P} := \mathbf{P}^F + \mathbf{P}^S - \frac{\rho^F \rho^S}{\rho} \mathbf{F}^S \mathbf{X}'^F \otimes \mathbf{X}'^F,$$

$$\rho\varepsilon := \rho^F \varepsilon^F + \rho^S \varepsilon^S + \frac{1}{2}\frac{\rho^F \rho^S}{\rho} \mathbf{C}^S \cdot \left(\mathbf{X}'^F \otimes \mathbf{X}'^F\right),$$

$$\boxed{\mathbf{C}^S := \mathbf{F}^{ST} \mathbf{F}^S,}$$

Right Cauchy-Green deformation tensor

$$\mathbf{Q} := \mathbf{Q}^F + \mathbf{Q}^S + \frac{\rho^F \rho^S}{\rho}\left(\varepsilon^F - \varepsilon^S\right)\mathbf{X}'^F +$$

$$+ \frac{\rho^F \rho^S}{\rho}\left(\frac{1}{\rho^S}\mathbf{P}^{ST} - \frac{1}{\rho^F}\mathbf{P}^{FT}\right)\mathbf{F}^S \mathbf{X}'^F + \frac{1}{2}\left(\rho^S - \rho^F\right)\frac{\rho^F \rho^S}{\rho^2} \mathbf{X}'^F \cdot \mathbf{C}^S \mathbf{X}'^F\, \mathbf{X}'^F.$$

Constitutive relations for poroelastic materials

Constitutive variables:

$$\mathrm{C}^{(1)}: \quad \Re := \left\{\rho^F, n, \mathbf{F}^S, \mathbf{X}'^F, \theta, \mathbf{G}\right\}, \quad \mathbf{G} := Grad\,\theta.$$

simple model

$$\mathrm{C}^{(2)}: \quad \Re := \left\{\rho^F, n, \mathbf{N}, \mathbf{F}^S, \mathbf{X}'^F, \theta, \mathbf{G}\right\}, \quad \mathbf{N} := Grad\,n.$$

second gradient model

(precursor of the Biot‘s model)

Constitutive quantities:

$$\mathrm{F} := \left\{\mathbf{P}^F, \mathbf{P}^S, \hat{\mathbf{p}}, \mathbf{J}, \hat{n}, \varepsilon, \mathbf{Q}\right\}$$

We assume the lack of mass sources!

Constitutive relations (local and homogeneous!):

$$\mathrm{F} = \mathrm{F}(\Re).$$

Restrictions:

1) Thermodynamical admissibility – the second law of thermodynamics,
2) Material objectivity,
3) Isotropy.

Second law of thermodynamics

Partial balance equations + constitutive relations = field equations.

For all solutions of field equations the following entropy inequality

$$\mathsf{H}^* = \frac{\partial\left(\rho^S \eta^S\right)}{\partial t} + Div\,\mathbf{H}^S + \frac{\partial\left(\rho^F \eta^F\right)}{\partial t} + Div\left(\rho^F \eta^F \mathbf{X}'^F + \mathbf{H}^F\right) \geq 0,$$

$$\eta^S = \eta^S(\mathfrak{R}), \quad \eta^F = \eta^F(\mathfrak{R}), \quad \mathbf{H}^S = \mathbf{H}^S(\mathfrak{R}), \quad \mathbf{H}^F = \mathbf{H}^F(\mathfrak{R}),$$

must be satisfied identically.

I-Shih Liu Theorem $\Longrightarrow$

$$\forall_{\text{fields}} \quad \mathsf{H}^* - \lambda^S \mathsf{B}^{\rho^S} - \lambda^F \mathsf{B}^{\rho^F} - \mathbf{\Lambda}^S \cdot \mathsf{B}^{\mathbf{v}^S} - \mathbf{\Lambda}^F \cdot \mathsf{B}^{\mathbf{v}^F} - \lambda^n \mathsf{B}^n - \Lambda^\varepsilon \mathsf{B}^\varepsilon \geq 0.$$

$$\lambda^S = \lambda^S(\mathfrak{R}), \quad \lambda^F = \lambda^F(\mathfrak{R}), \quad \mathbf{\Lambda}^S = \mathbf{\Lambda}^S(\mathfrak{R}), \quad \lambda^n = \lambda^n(\mathfrak{R}), \quad \Lambda^\varepsilon = \Lambda^\varepsilon(\mathfrak{R}).$$

Details of exploitation:

Wilmanski, K.; *Multicomponent models in geophysics*, in: J. Majewski, R. Teisseyre (eds.), Ea*rthquake Thermodynamics and Phase Transformations in the Earth's Interior*, 567-655, Academic Press (2001), Results for the class of isothermal processes $\theta(\mathbf{X},t) = const.$

under the assumption*

$$\frac{\partial \psi^S}{\partial \mathbf{X}'^F} = \frac{\partial \psi^F}{\partial \mathbf{X}'^F} = 0, \quad \psi^S := \varepsilon^S - \theta\eta^S, \quad \psi^F := \varepsilon^F - \theta\eta^F.$$

Helmholtz free energies are constitutive quantities:

$$\psi^S = \psi^S(\mathfrak{R}), \quad \psi^F = \psi^F(\mathfrak{R}),$$

Residual inequality (dissipation)

$$\mathrm{D} := \hat{\mathbf{p}} \cdot \mathbf{F}^S \mathbf{X}'^F - \left(\rho^S \frac{\partial \psi^S}{\partial n} + \rho^F \frac{\partial \psi^F}{\partial n} \right) \hat{n} \geq 0.$$

Thermodynamical equilibrium: $\mathrm{D}\big|_E = 0.$

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*without this assumption models seem to include fluctuations of microscopic kinetic energies leading to a description of tortuosity. See:

Hutter, K., Kubik, J., Kosinski, W. (to appear).

Isotropy

Dependence of scalar constitutive quantities on invariants:

$$I := tr\,\mathbf{C}^S, \quad II := \tfrac{1}{2}\left(I^2 - tr\,\mathbf{C}^{S2}\right), \quad III := J^{S2} = \det \mathbf{C}^S,$$

$$IV := \mathbf{X}'^F \cdot \mathbf{X}'^F, \quad V := \mathbf{X}'^F \cdot \mathbf{C}^S \mathbf{X}'^F, \quad VI := \mathbf{X}'^F \cdot \mathbf{C}^{S2} \mathbf{X}'^F.$$

rather than on $\mathbf{F}^S, \mathbf{X}'^F$. Hence

$$\psi^S := \psi^S\left(\rho^S, \rho^F, I, II, III, IV, V, VI, n\right),$$
$$\psi^F := \psi^F\left(\rho^S, \rho^F, I, II, III, IV, V, VI, n\right),$$
$$\mathbf{J} = \Gamma \mathbf{X}'^F + \Lambda Grad\, n, \quad \hat{\mathbf{p}} = \pi \mathbf{F}^S \mathbf{X}'^F - N \mathbf{F}^{S-1} Grad\, n, \quad \text{etc.}$$

Class $C^{(1)}$ under the assumption of linear deviation from the thermodynamical equilibrium $\hat{n}\big|_E = 0, \quad \mathbf{X}'^F\big|_E = 0$:

$$\mathbf{P}^S = \rho^S \frac{\partial \psi^S}{\partial \mathbf{F}^S}, \quad \mathbf{P}^F = -\rho^{F2} \frac{\partial \psi^F}{\partial \rho^F} \mathbf{F}^{S-T},$$

$$\psi^S = \psi^S\left(\mathbf{F}^S\right), \quad \psi^F = \psi^F\left(\rho^F J^{S-1}\right), \quad J^S := \det \mathbf{F}^S.$$

$$\hat{\mathbf{p}} = \pi \mathbf{F}^S \mathbf{X}'^F, \quad \mathbf{J} = n_E \mathbf{X}'^F, \quad \hat{n} = -\frac{n - n_E}{\tau}, \quad n_E = n_E\left(\frac{\rho^F}{\rho^S}\right).$$

Cauchy stresses:

$$\mathbf{T}^S = \rho_t^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} \mathbf{F}^{ST}, \quad \mathbf{T}^F = -p^F \mathbf{1}, \quad p^F := \rho_t^{F2} \frac{\partial \psi^F}{\partial \rho_t^F},$$

$$\psi^S = \psi^S\left(\mathbf{F}^S\right), \quad \psi^F = \psi^F\left(\rho_t^F\right), \quad \rho_t^S := \rho^S J^{S-1}, \quad \rho_t^F := \rho^F J^{S-1},$$

Hence there is no coupling between components through partial stresses – simple mixture.

Class $C^{(2)}$ under the same assumption and the linearity of dependence on the porosity gradient . Then the only dependence on this gradient:

$$\hat{\mathbf{p}} = \pi \, \mathbf{F}^{S} \mathbf{X}'^{F} - N \mathbf{F}^{S-1} Grad\, n.$$

Cauchy stresses

$$\mathbf{T}^{S} = \rho_t^{S} \left[J^{S} \frac{\partial \psi^{S}}{\partial J^{S}} + 2 \left(\frac{\partial \psi^{S}}{\partial I} + I \frac{\partial \psi^{S}}{\partial II} \right) \mathbf{B}^{S} - 2 \frac{\partial \psi^{S}}{\partial II} \mathbf{B}^{S2} \right] - \underline{n_0 N \frac{\rho_t^{F}}{\rho_0^{F}} \mathbf{1}},$$

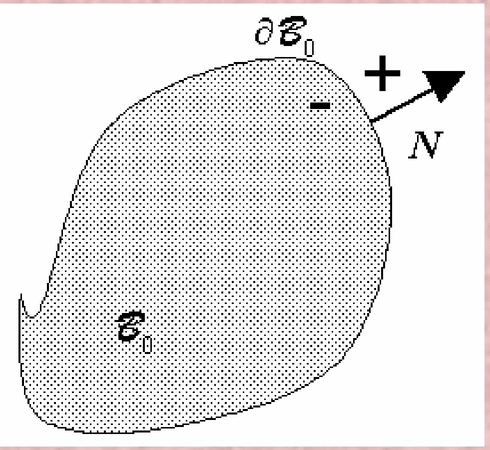
$$\mathbf{T}^{F} = -\left(\rho_t^{F2} \frac{\partial \psi^{F}}{\partial \rho_t^{F}} + n_0 N \frac{\rho_t^{F}}{\rho_0^{F}} \right) \mathbf{1}, \quad \mathbf{B}^{S} := \mathbf{F}^{S} \mathbf{F}^{ST},$$ Left Cauchy-Green deformation

$$\psi^{S} = \psi^{S}\left(I, II, J^{S}, \rho_t^{F}\right), \quad \psi^{F} = \psi_{id}^{F}\left(\rho_t^{F}\right) - \frac{1}{\rho_t^{F}} \int_{1}^{J^{S}} \frac{N\left(\rho_t^{F}, \xi\right)}{\xi} d\xi, \quad \text{e.g.} \quad N = \frac{N_0}{J^{S}}.$$

Linearization yields Biot‘s coupling of partial stresses.

Boundary conditions

We need two vectorial boundary conditions (momentum equations) and a boundary condition for the porosity equation (?). We considers solely the boundary material with respect to the skeleton: ∂B_0



1) Boundary condition for the total stress

$$\left.\left(\mathbf{P}^S + \mathbf{P}^F\right)\mathbf{N}\right|_{\partial B_0} \approx \mathbf{t}_{ext}.$$

2) Flow through a permeable boundary

$$\left.\rho^F \mathbf{X}'^F \cdot \mathbf{N} - \alpha_0 \left(\frac{p^{F-}}{n^-} - \frac{p^{F+}}{n^+} \right)\right|_{\partial B_0} = 0.$$

3) For the ideal fluid – lack of the tangential flow*

$$\left.\mathbf{X}'^F - \mathbf{X}'^F \cdot \mathbf{N}\,\mathbf{N}\right|_{\partial B_0} = 0.$$

4) There seems to be no need for a boundary condition for porosity**

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*see: Beavers, G. S., Joseph, D. D.; *Boundary conditions at a naturally permeable wall,* J. Fluid Mech., **30** (1), 197-207 (1967).

**see: Radkevich, E. V., Wilmanski, K.; *A Riemann problem for poroelastic materials with the balance equation for porosity*, WIAS – Preprints, #593, 594 (2000).

Linear model of poroelastic saturated materials without mass exchange

$$\frac{\partial \rho_t^F}{\partial t} + div\left(\rho_t^F \mathbf{v}^F\right) = 0, \quad \frac{\partial \rho_t^S}{\partial t} + div\left(\rho_t^S \mathbf{v}^S\right) = 0, \quad \mathbf{x} \in \chi(\mathrm{B}_0, t),$$

$$\rho_t^S \frac{\partial \mathbf{v}^S}{\partial t} = div\left[\lambda^S e \mathbf{1} + 2\mu^S \mathbf{e}^S + \underline{\underline{\beta \Delta_n}} \mathbf{1} + \underline{\underline{n_0 N\varepsilon}} \mathbf{1}\right] + \pi\left(\mathbf{v}^F - \mathbf{v}^S\right) - \underline{\underline{N grad\, n}},$$

$$\rho_t^F \frac{\partial \mathbf{v}^F}{\partial t} = grad\left[\rho_0^F \kappa\varepsilon + \underline{\underline{n_0 N e}} - \underline{\underline{\beta \Delta_n}}\right] - \pi\left(\mathbf{v}^F - \mathbf{v}^S\right) + \underline{\underline{N grad\, n}},$$

$$\frac{\partial \Delta_n}{\partial t} + n_0 div\left(\mathbf{v}^F - \mathbf{v}^S\right) = -\frac{\Delta_n}{\tau}, \qquad e := tr\,\mathbf{e}^S, \quad \varepsilon := \frac{\rho_0^F - \rho_t^F}{\rho_0^F}, \quad \Delta_n := n - n_E,$$

for small deformations, i.e.

$$\mathbf{e}^S := \tfrac{1}{2}\left(\mathbf{1} - \mathbf{B}^{S-1}\right), \quad \left\|\mathbf{e}^S\right\| << 1, \quad \varepsilon << 1, \quad n_E \approx n_0\left(1 + e - \varepsilon\right).$$

Special cases:

$N = 0 \quad \Rightarrow \quad$ C $^{(1)}$ - model,

$\beta = 0, \quad n = n_E \quad \Rightarrow \quad$ C $^{(2)}$ - (Biot‘s) model; the original notation:

$$K := \lambda^S - n_0 N + \tfrac{2}{3}\mu^S, \quad Q := 2n_0 N, \quad R := \rho_0^F \kappa - n_0 N,$$

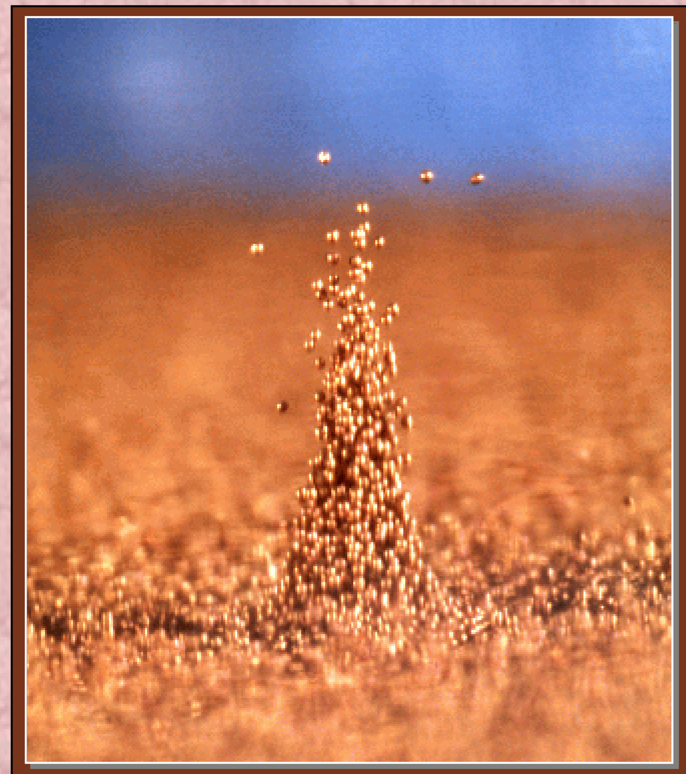
$$\mathbf{T}^S = Ke\mathbf{1} + 2\mu^S dev\,\mathbf{e}^S + Q\varepsilon\mathbf{1}, \quad dev\,\mathbf{e}^S := \mathbf{e}^S - \tfrac{1}{3}e\mathbf{1},$$

$$\mathbf{T}^F = -p^F\mathbf{1}, \quad p^F = -R\varepsilon - Qe.$$

The main problem of the linear modeling: dependence of macroscopic material parameters on the initial porosity n_0. For granular materials –
- attempts to derive it from simple homogeneous experiments.

Oscillon

Paul Umbanhowar (Northwestern Univ.)



A few remarks on sound waves in linear poroelastic materials

Two modes of longitudinal waves: P1 and P2 (Biot, second sound)

simple model:

$$c_{P1}^2 := \frac{\lambda^S + 2\mu^S}{\rho_0^S}, \quad c_{P2}^2 := \kappa.$$

Biot's model:

$$c_{P1}^2 := \frac{\lambda^S + 2\mu^S - \frac{1}{2}Q}{\rho_0^S}, \quad c_{P2}^2 := \frac{R - \frac{1}{2}Q}{\rho_0^F}.$$

One mode of shear wave: S

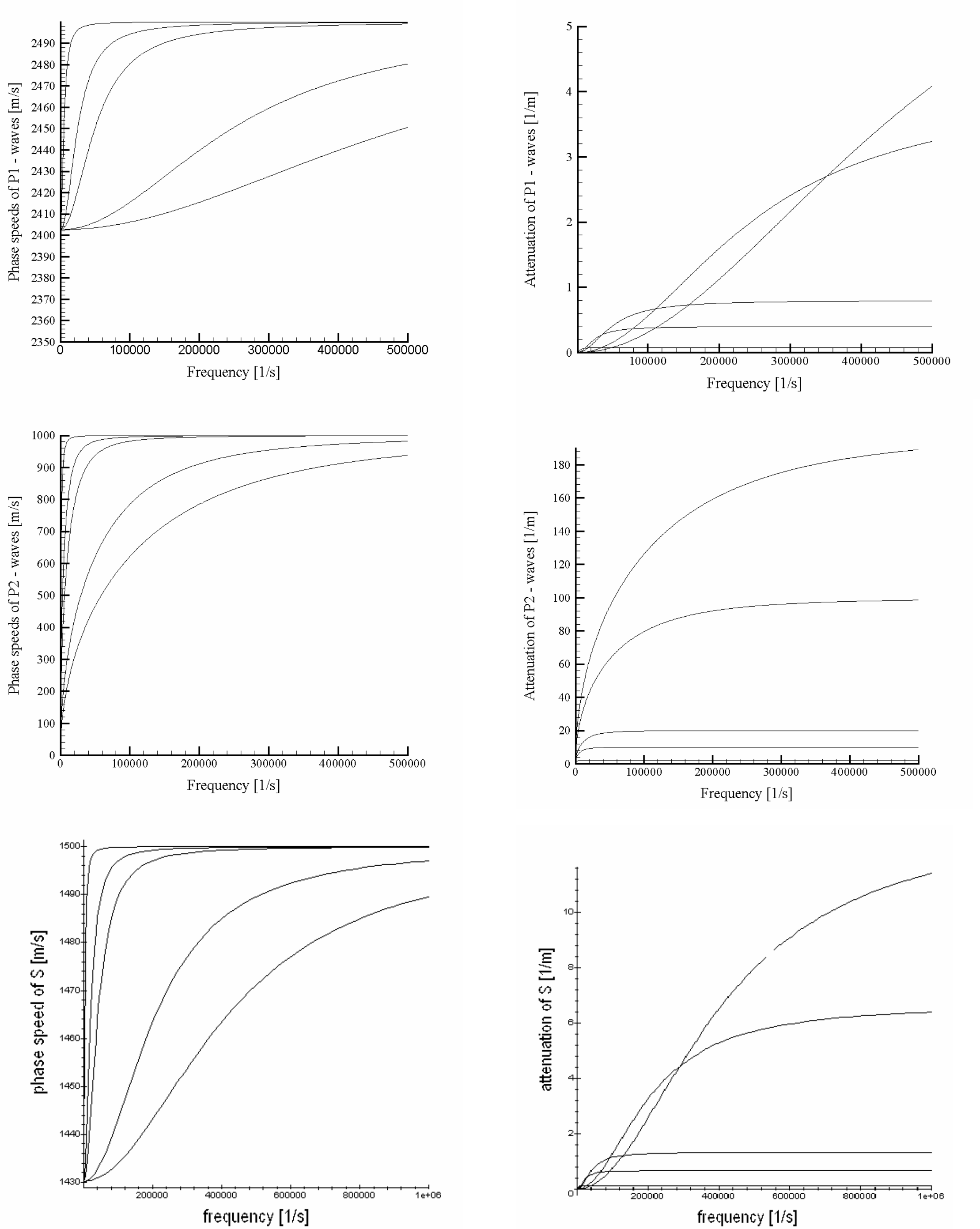
$$c_S^2 := \frac{\mu^S}{\rho_0^S}.$$

Monochromatic waves:

$$P1: \quad c_{oP1} := \lim_{\omega\to 0}\frac{\omega}{\operatorname{Re} k} = \sqrt{\frac{\lambda^S + 2\mu^S + \kappa\rho_0^F}{\rho_0^S + \rho_0^F}}, \quad \lim_{\omega\to\infty}\frac{\omega}{\operatorname{Re} k} = c_{P1},$$

$$P2: \quad c_{oP2} := \lim_{\omega\to 0}\frac{\omega}{\operatorname{Re} k} = 0, \quad \lim_{\omega\to\infty}\frac{\omega}{\operatorname{Re} k} = \sqrt{\kappa} = c_{P2}.$$

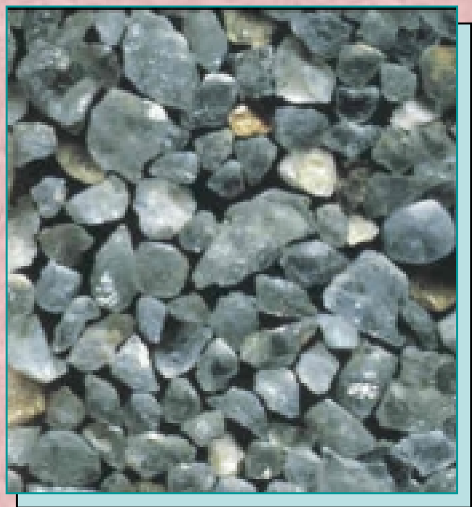
$$c_{oS} := \lim_{\omega\to 0}\frac{\omega}{\operatorname{Re} k} = \sqrt{\frac{\mu^S}{\rho_0^S + \rho_0^F}}, \quad \lim_{\omega\to\infty}\frac{\omega}{\operatorname{Re} k} = \sqrt{\frac{\mu^S}{\rho_0^S}} = c_S.$$



Phase speeds and attenuations

of P1, P2 , and S– waves as functions of the frequency ω for c_{P1} =2500 m/s, c_{P2} =1000 m/s, c_S =1500 m/s, ρ_0^S =2500 kg/m^3, ρ_0^F =250 kg/m^3, permeabilities π =10^6, 5$\times$10^6, 10^7, 5$\times$10^7, 10^8 kg/m^{3}s.

Micro-macro-transitions for granular materials



Within models of saturated poroelastic materials we can expect to obtain an information on a dependence of material parameters $\lambda^S, \mu^S, \kappa, \pi$ for the simple model or K, μ^S, R, Q, π for Biot's model on real microscopic compressibilities and the initial porosity n_0.

In both models the shear modulus of granular materials μ^S must depend in a nonlinear way at least on a confining pressure. Therefore it is further considered to be known.

We seek

$$\left.\begin{aligned} \lambda^S &= \lambda^S\left(K_S, K_F, n_0; \mu^S(n_0, \ldots)\right) \\ \kappa &= \kappa\left(K_S, K_F, n_0; \mu^S(n_0, \ldots)\right) \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} c_{P1} &= c_{P1}\left(K_S, K_F, \mu^S, \rho^{SR}, \rho^{FR}, n_0\right), \\ c_{P2} &= c_{P2}\left(K_S, K_F, \mu^S, \rho^{SR}, \rho^{FR}, n_0\right), \\ c_R &= c_R\left(K_S, K_F, \mu^S, \rho^{SR}, \rho^{FR}, n_0\right), \end{aligned}\right.$$

where K_S, K_F are real (microscopic) compressibility moduli.

Geometrical compatibility of micro and macrodescriptions:

Notation: M – homogeneous microstructure,

$V := \text{volume}\,(\mathrm{M}),\quad V^F$ - volume of voids, $\quad V^S := V - V^F.$

e, ε - macroscopic volume changes

$$e^R := \frac{V^S - V_0^S}{V_0^S},\quad \varepsilon^R := \frac{V^F - V_0^F}{V_0^F}$$ - microscopic volume changes

a) Porosity relations ($n=n_E$!):

macro
$$n = n_0 \frac{\rho^F}{\rho_0^F}\frac{\rho_0^S}{\rho^S} = n_0 \frac{1+e}{1+\varepsilon} \approx n_0\left(1+e-\varepsilon\right),$$

micro
$$n = \frac{V^F}{V^F + V^S} = \frac{n_0\left(1+\varepsilon^R\right)}{n_0\left(1+\varepsilon^R\right) + \left(1-n_0\right)\left(1+e^R\right)} \approx$$
$$\approx n_0 + n_0\left(1-n_0\right)\left(\varepsilon^R - e^R\right).$$

Hence

$$\frac{1+e}{1+\varepsilon} = \frac{1+\varepsilon^R}{n_0(1+\varepsilon^R)+(1-n_0)(1+e^R)}.$$

For small deformations:

$$e-\varepsilon = (1-n_0)\left(\varepsilon^R - e^R\right).$$

b) Mass conservation of the skeleton:

$$\rho^{SR} = \rho_0^{SR}\left(1+e^R\right)^{-1} \quad \text{and} \quad \rho^S = (1-n)\rho^{SR}.$$

It follows

$$1-n_0\frac{1+e}{1+\varepsilon} = \frac{1+e^R}{1+e}.$$

For small deformations:

$$(1-n_0)e^R = (1-2n_0)e + n_0\varepsilon.$$

Resultant geometrical compatibility relations

$$e = n_0 \varepsilon^R + (1 - n_0) e^R, \quad \varepsilon = -(1 - 2n_0)\varepsilon^R + 2(1 - n_0) e^R. \tag{1}$$

Dynamical compatibility relations:

$$p^S = (1 - n_0) p^{SR}, \quad p^F = n_0 p^{FR}. \tag{2}$$

Equilibrium conditions (p' – excess pressure):

macro: $p^F + p^S = p',$ micro: $n_0 p^{FR} + (1 - n_0) p^{SR} = p',$

Constitutive relations:

macro (Wilmanski):

$$p^S = -Ke, \quad K := \lambda^S + \frac{2}{3}\mu^S,$$
$$p^F = -R\varepsilon, \quad R := \kappa \rho_0^F.$$

macro (Biot):

$$p^S = -Ke - Q\varepsilon,$$
$$p^F = -R\varepsilon - Qe.$$

micro:

$$p^{SR} = -K_S e^R, \quad p^{FR} = -K_F \varepsilon^R.$$

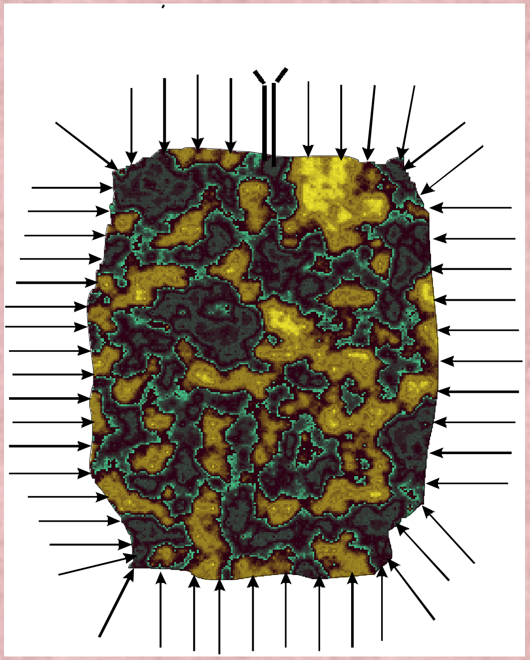
Relations (1) and (2) cannot be fulfilled simultaneously !

For instance in a drained jacketed experiment we have

$$p^{FR} = 0 \quad \Rightarrow \quad p^{F} = 0$$

$$\downarrow \qquad\qquad \downarrow$$

$$\varepsilon^{R} = 0 \qquad\qquad \varepsilon = 0$$



Then geometrical compatibility conditions (1) yield

$$e = e^{R} = 0$$

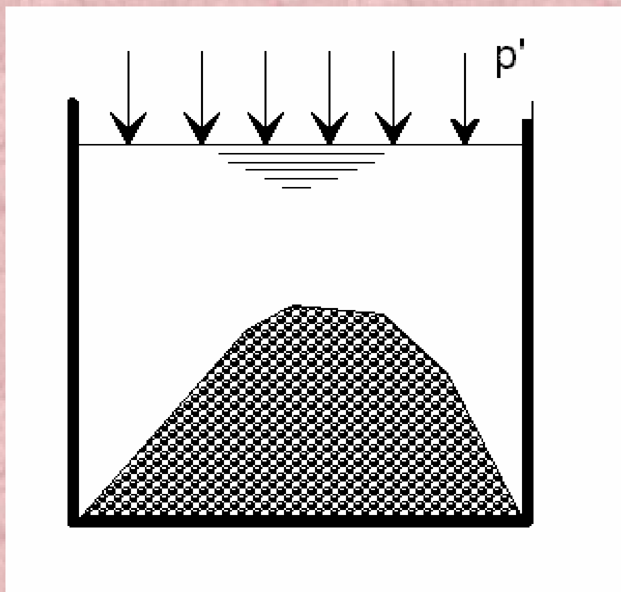
which cannot hold!

Such inconsistencies are known: for example Taylor and Sachs approach to the micro-macro-transition in crystal plasticty. Each approach yields solely an upper or lower estimate of macroscopic material parameters.

We present results for geometrically consistent approach.

Gedanken (elementary) experiments

a) unjacketed

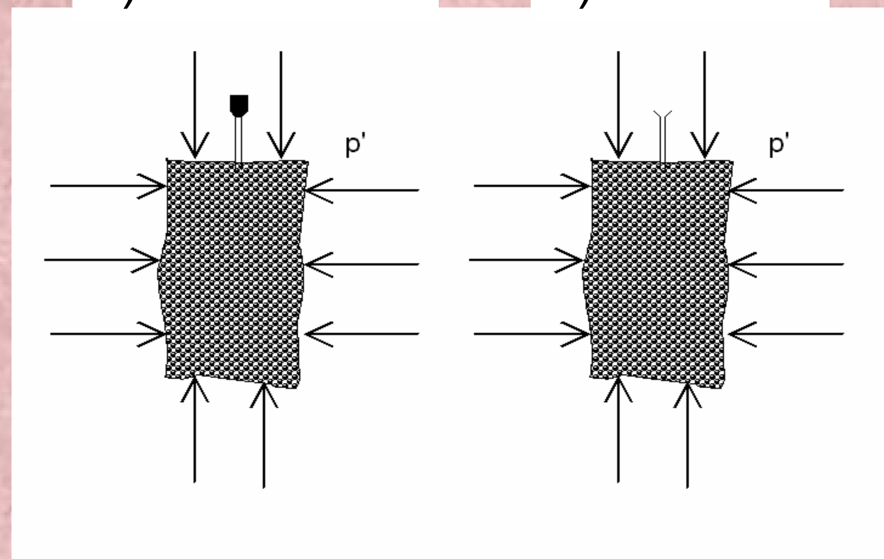


$$p^F = n_0 p'$$

jacketed

b) undrained

c) drained



$$\frac{d\rho^F V}{dt} = 0$$

$$p^F = 0$$

a) unjacketed:

$$p^F = n_0 p' \quad \Rightarrow \quad p^S = (1-n_0)p', \qquad \text{(instead of } p^{FR} = p'\text{)}$$

$$n_0 p^{FR} + (1-n_0)p^{SR} = p'.$$

Hence $$e = -(1-n_0)\frac{p'}{K}, \quad \varepsilon = -n_0 \frac{p'}{R}.$$

It follows by substitution in (1) and the microequilibrium:

$$\begin{pmatrix} n_0 & 1-n_0 & \dfrac{1-n_0}{K} \\ -(1-2n_0) & 2(1-n_0) & \dfrac{n_0}{R} \\ -n_0 K_F & -(1-n_0)K_S & -1 \end{pmatrix} \begin{pmatrix} \varepsilon^R \\ e^R \\ p' \end{pmatrix} = 0,$$

for all p'. Hence

$$n_0 \left\{ \frac{n_0}{R} - \frac{2(1-n_0)}{K} \right\} (K_S - K_F) + (1-n_0)\frac{K_S}{K} = 1. \qquad \text{(unj.)}$$

b) jacketed undrained: $\frac{d\rho^F}{dt} = 0 \quad \Rightarrow \quad e = \varepsilon = e^R = \varepsilon^R.$

It follows

$$K + R = n_0 K_F + (1 - n_0) K_S. \qquad \text{(jud.)}$$

c) jacketed drained: $p^F = 0 \quad \Rightarrow \quad \varepsilon = 0,\ e = \frac{1}{2}\varepsilon^R,\ e^R = \frac{2(1-n_0)}{1-2n_0}\varepsilon^R.$

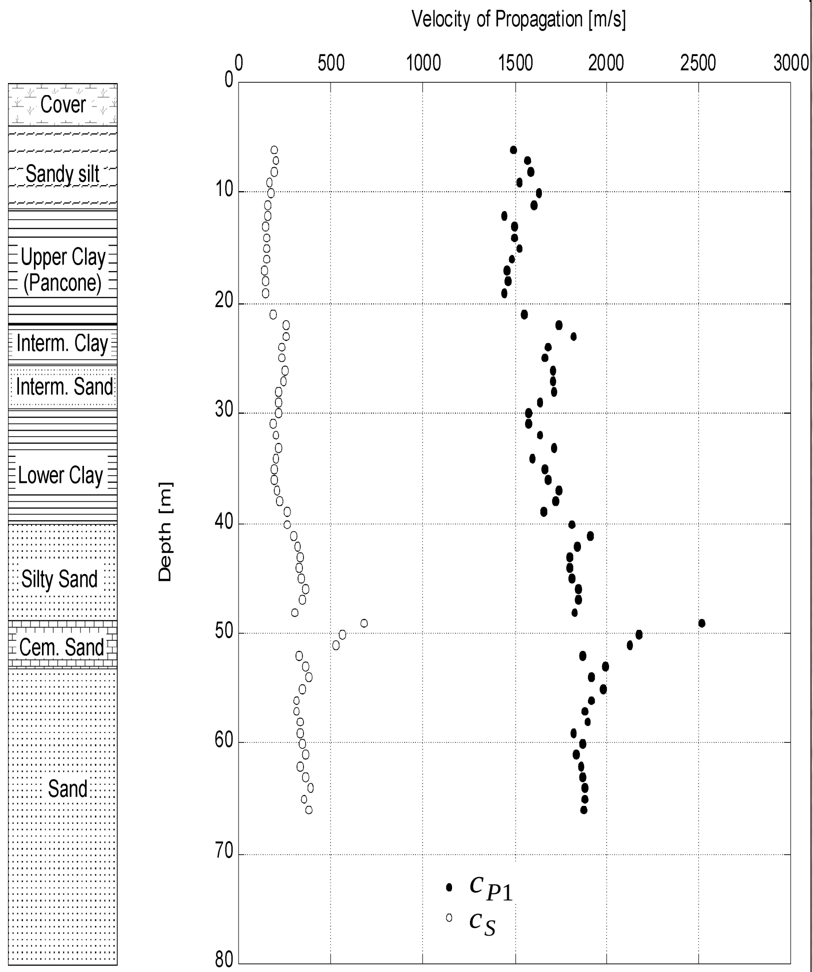
It follows

$$K = K_S - 2n_0 (K_S - K_F). \qquad \text{(jd.)}$$

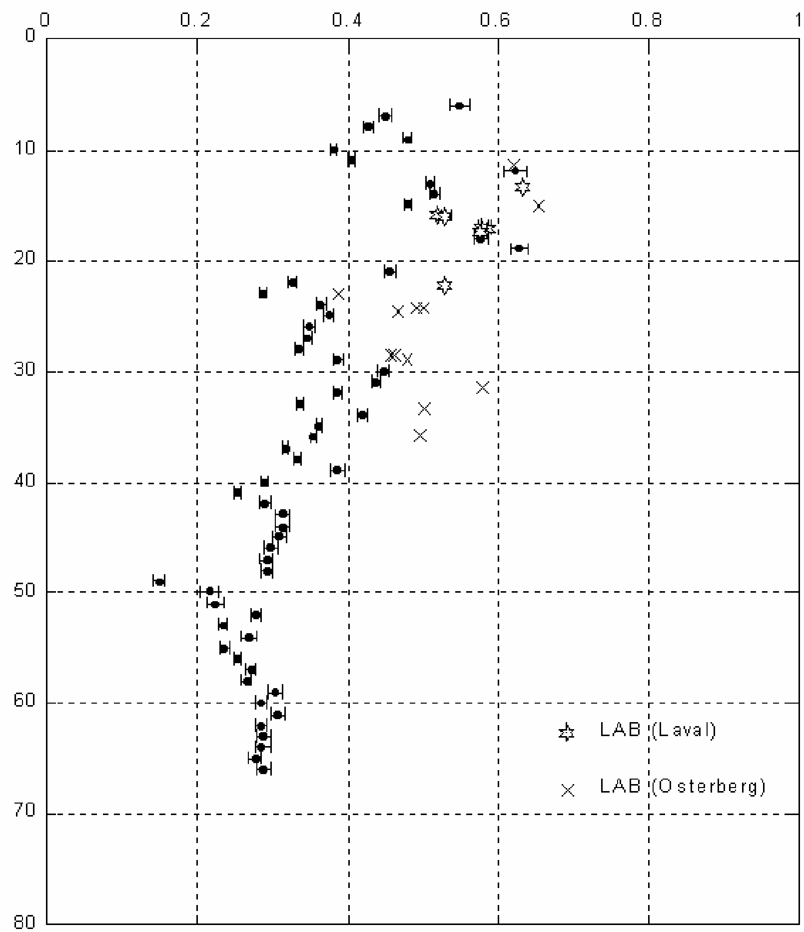
Equations (unj.), (jud.), and (jd.) cannot be fulfilled simultaneously.
Either we have to ignore one of the experiments or we have to work with a model containing an additional material parameter (coupling) e.g. Q-constant of the Biot‘s model.

For such a model the results for the following practical example were obtained.

Comparison of experimental and theoretical results



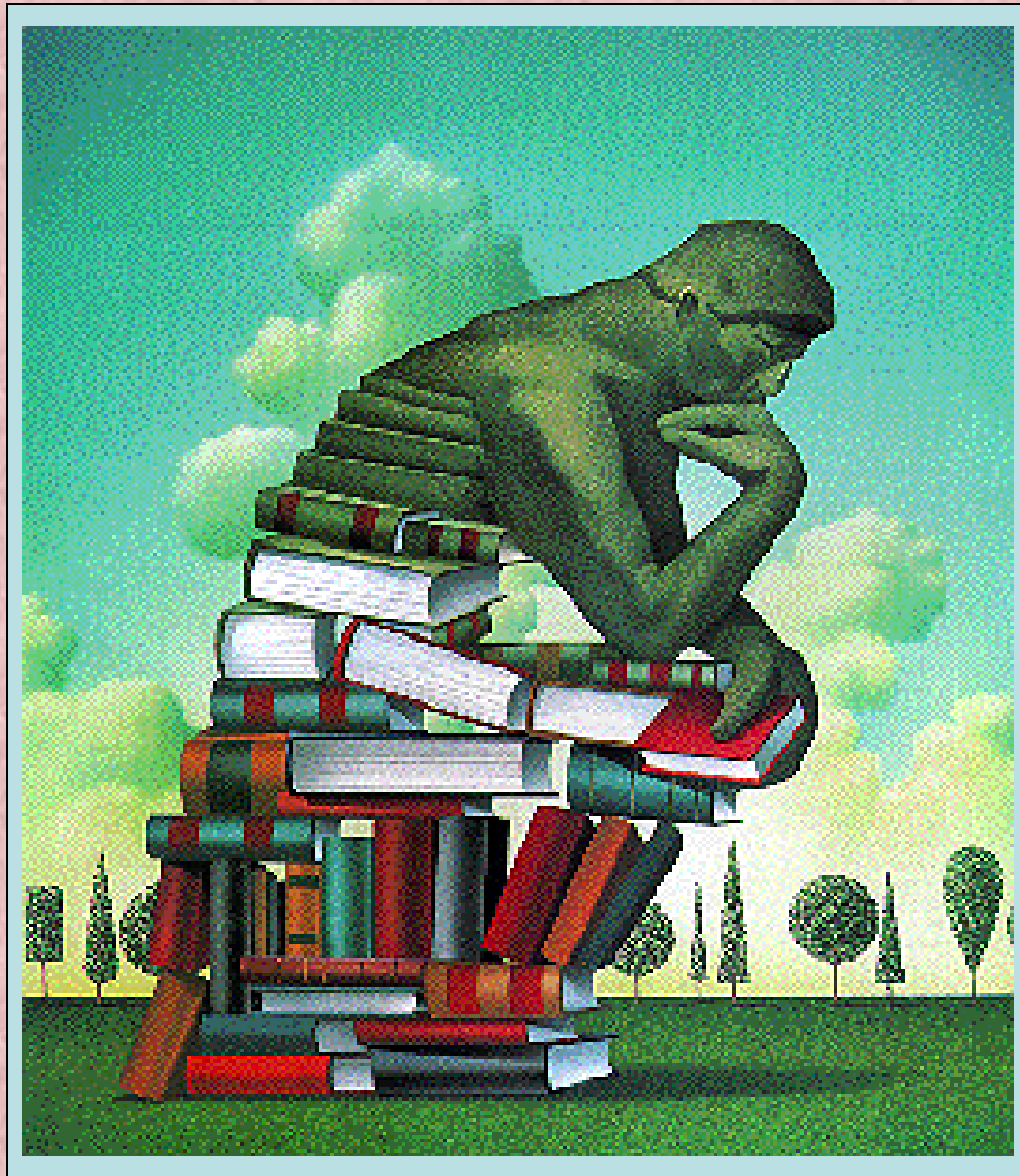
Soil profile and charts of speeds c_{P1} and c_S at Pisa site



Porosity predicted at Pisa site for measured speeds of bulk waves. Comparison with data by Laval and Osterberg laboratory tests

Foti,S., Lai, C., Lancellotta, R.; *Porosity of fluid-saturated porous media from measured seismic wave velocities,* Géotechnique (2002, to appear).

Concluding remarks



Some results for $C^{(1)}$ - model

1. Propagation of sound waves in nonlinear poroelastic materials (nonlinear elasticity and nonlinear gas law without dependence on porosity)
2. Steady state flow through a poroelastic cylinder (Signiorini elasticity) (nonlinear elasticity with porosity dependence)
3. Sound waves in linear poroelastic materials with a nonlinear fluid – critical amplitude (nonlinear gas law without dependence on porosity)
4. Monochromatic waves in linear poroelastic materials (small deformations, no dependence on porosity)
5. Steady state flows in poroelastic materials with adsorbtion (Langmuir; linear elasticity, nonlinear mass ecxchange)
6. Linear surface waves in poroelastic materials – contact with vacuum or fluid (small deformations, no dependence on porosity)
7. Relaxation and stability of steady state flows – 1D flows, 2D disturbances (small deformation, nonlinear mass exchange)
8. Structural instability, piping in the model extended by a nonlinear momentum source (small deformation, nonlinear dependence on porosity gradient)
9. Shock and soliton-like solutions of Riemann problems for poroelastic materials (small deformation, nonlinear dependence on porosity)

Some open problems

1. Unsaturated materials – construction of the model, phase transformations, coupling with acoustic waves

2. Selfconsistent micro-macro-transitions for granular materials

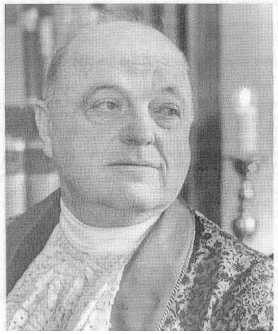
3. Nonlinear waves – in particular nonlinear surface waves in two-component materials

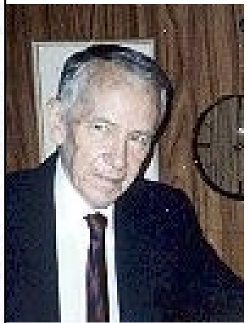
4. Mass exchange processes with the change of morphology

5. Extension of description of microstructure – anisotropy, tortuosity, fluctuations of microscopic kinetic energy.

Mixtures, porous materials and waves

Gallery


C. A.Truesdell


M. A. Biot

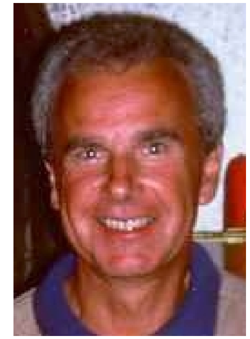

I. Müller


I-Shih Liu


K. Hutter


O. Coussy

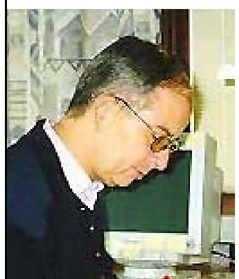

A. Cheng

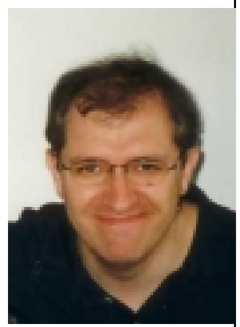

B. A. Schefler


G. Maugin


N. Kirchner


B. Albers


D. Kolymbas

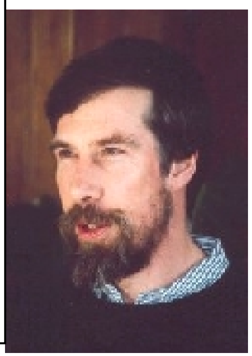

Carlo Lai


R. Lancellotta


Glenn J. Rix


E. Kausel

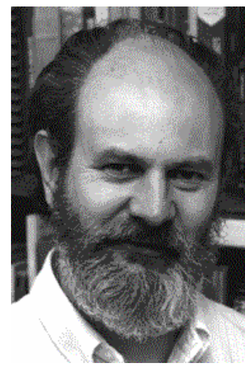

J. Berryman


David M. Wood


Jacob Bear


K. Wilmanski


M. Kaviany


P. G. Richards


Keiiti Aki

Ray Bowen !

Pieter Brueghel the Elder (1560), *Temperantia*
- question of education, reading, writing, calculation, geometry, singing lecturing, astronomy, geography.