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Linear Sound Waves in Poroelastic Materials: Simple Mixture vs. Biot's Model

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Contents:

Unknown fields of the linear model

$$\left\{ \rho^F, \mathbf{v}^S, \mathbf{v}^F, \mathbf{e}^S, \cancel{n} \right\}$$

Biot

- partial mass density of the fluid
- velocity of the skeleton
- velocity of the fluid
- Almansi-Hamel deformation tensor of the skeleton
- **porosity**

Balance equations – linear model

$$\frac{\partial \rho^S}{\partial t} + \rho_0^S \, div \, \mathbf{v}^S = 0, \quad \frac{\partial \rho^F}{\partial t} + \rho_0^F \, div \, \mathbf{v}^F = 0,$$

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = div \, \mathbf{T}^S + \hat{\mathbf{p}} + \rho^S \mathbf{b}^S, \quad \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = div \, \mathbf{T}^F - \hat{\mathbf{p}} + \rho^F \mathbf{b}^F,$$

$$\frac{\partial (n - n_E)}{\partial t} + \Phi \, div \, (\mathbf{v}^F - \mathbf{v}^S) = -\frac{n - n_E}{\tau}.$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = sym \, grad \, \mathbf{v}^S.$$

Linear constitutive relations

Partial stresses

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S e\mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon\mathbf{1} - \mathrm{N}(n-n_0)\mathbf{1},$$

$$\mathbf{T}^F = -p_{\mathrm{int}}^F \mathbf{1} + \mathrm{N}(n-n_0)\mathbf{1}, \quad p_{\mathrm{int}}^F = p_0^F - (\rho_0^F \kappa\varepsilon + Qe),$$

$$e := tr\,\mathbf{e}^S, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F}.$$

Porosity

$$n_E = n_0(1+\delta e), \quad \Phi = const.$$

material constants

Linear field equations

Biot

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \lambda^S grad\, e + 2\mu^S div\, \mathbf{e}^S + Q grad \varepsilon + \pi(\mathbf{v}^F - \mathbf{v}^S) - \rho_{12}\left(\frac{\partial \mathbf{v}^F}{\partial t} - \frac{\partial \mathbf{v}^S}{\partial t}\right) - \cancel{\mathrm{N} grad\, n},$$

$$\rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = Q grad\, e + \rho_0^F \kappa\, grad \varepsilon - \pi(\mathbf{v}^F - \mathbf{v}^S) + \rho_{12}\left(\frac{\partial \mathbf{v}^F}{\partial t} - \frac{\partial \mathbf{v}^S}{\partial t}\right) + \cancel{\mathrm{N} grad\, n},$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = sym\, grad\, \mathbf{v}^S, \quad \frac{\partial \varepsilon}{\partial t} = div\, \mathbf{v}^F, \quad \frac{\partial}{\partial t}\left[n - n_0\delta e + \Phi(e-\varepsilon)\right] = -\frac{n - \cancel{n_0 -} n_0\delta e}{\tau}.$$

Linear constitutive relations

Partial stresses

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S e\mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon\mathbf{1} - \mathrm{N}(n - n_0)\mathbf{1},$$

$$\mathbf{T}^F = -p_{\mathrm{int}}^F \mathbf{1} + \mathrm{N}(n - n_0)\mathbf{1}, \quad p_{\mathrm{int}}^F = p_0^F - (\rho_0^F \kappa\varepsilon + Qe),$$

$$e := tr\mathbf{e}^S, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F}.$$

Porosity

$$n_E = n_0(1 + \delta e), \quad \Phi = const.$$

material constants

Linear field equations

Simple mixture

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \lambda^S grad\, e + 2\mu^S div\, \mathbf{e}^S + \cancel{Q grad \varepsilon} + \pi(\mathbf{v}^F - \mathbf{v}^S) - \cancel{\rho_{12}\left(\frac{\partial \mathbf{v}^F}{\partial t} - \frac{\partial \mathbf{v}^S}{\partial t}\right)} - \cancel{\mathrm{N} grad\, n},$$

$$\rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = \cancel{Q grad\, e} + \rho_0^F \kappa\, grad \varepsilon - \pi(\mathbf{v}^F - \mathbf{v}^S) + \cancel{\rho_{12}\left(\frac{\partial \mathbf{v}^F}{\partial t} - \frac{\partial \mathbf{v}^S}{\partial t}\right)} + \cancel{\mathrm{N} grad\, n},$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = sym\, grad\, \mathbf{v}^S, \quad \frac{\partial \varepsilon}{\partial t} = div\, \mathbf{v}^F, \quad \frac{\partial}{\partial t}\left[n - n_0\delta e + \Phi(e - \varepsilon)\right] = -\frac{n - n_0 - n_0\delta e}{\tau}.$$

Gassmann relations:

$$N = 0$$

K, C, M – macroscopic compressibility moduli

$$K = \frac{(K_s - K_b)^2}{K_s(1+n_0\xi) - K_d} + K_d, \quad C = \frac{K_s(K_s - K_d)}{K_s(1+n_0\xi) - K_d},$$

$$M = \frac{K_s^2}{K_s(1+n_0\xi) - K_d},$$

$$Q = n_0(C - n_0 M).$$

$$\xi := \frac{K_s}{K_f} - 1.$$

$K_s, \quad K_f$ - true (microscopic) compressibility moduli,

K_d - drained compressibility modulus.

Berryman relation for tortuosity

$$\rho_{12} = \rho_0^F (1-a), \quad a = \tfrac{1}{2}\left(\tfrac{1}{n}+1\right), \quad i.e. \quad 1 \le a \le \infty.$$

It follows

$$\lambda^S + 2\mu^S = \frac{3(1-\nu)}{1+\nu}\left\{\frac{(K_s - K_d)^2}{\dfrac{K_s^2}{K_W} - K_d} + K_d\right\}, \quad \frac{1}{K_W} = \frac{1-n}{K_s} + \frac{n}{K_f},$$

$$\mu^S = \frac{3(1-2\nu)}{2(1+\nu)}\left\{\frac{(K_s - K_d)^2}{\dfrac{K_s^2}{K_W} - K_d} + K_d\right\}, \quad \rho_0^F \kappa = n^2 \frac{K_s^2}{\dfrac{K_s^2}{K_W} - K_d}, \qquad K_d = \frac{K_s}{1+50n}.$$

where ν - Poisson's number.

Changes of porosity

$$\frac{n-n_0}{n_0} = \delta e + \gamma(e-\varepsilon), \quad \delta := \frac{K_V - K}{n_0(K_s - K_f)}, \quad \gamma := \frac{\rho_0^F \kappa + Q - n_0 K_f}{n_0(K_s - K_f)},$$

$$K_V := (1-n_0)K_s + n_0 K_f, \quad K := \lambda^S + \tfrac{2}{3}\mu^S + \rho_0^F \kappa + 2Q.$$

Changes of porosity
- the solution of the porosity equation

$$\frac{\partial(n-n_E)}{\partial t}+\frac{n-n_E}{\tau}=-\Phi\, div(\mathbf{v}^F-\mathbf{v}^S), \quad n_E=n_0(1+\delta\, e).$$

Mass balance equations:

$$div\mathbf{v}^S=-\frac{\partial}{\partial t}\frac{\rho^S}{\rho_0^S}=\frac{\partial e}{\partial t}, \quad div\mathbf{v}^F=-\frac{\partial}{\partial t}\frac{\rho^F}{\rho_0^F}=\frac{\partial \varepsilon}{\partial t}.$$

Hence

$$\frac{n-n_0}{n_0}=\delta e+\frac{\Phi}{n_0}(e-\varepsilon)-\cancel{\frac{\Phi}{n_0\tau}\int_0^t (e-\varepsilon)(s)e^{-\frac{t-s}{\tau}}ds}.$$

memory effect

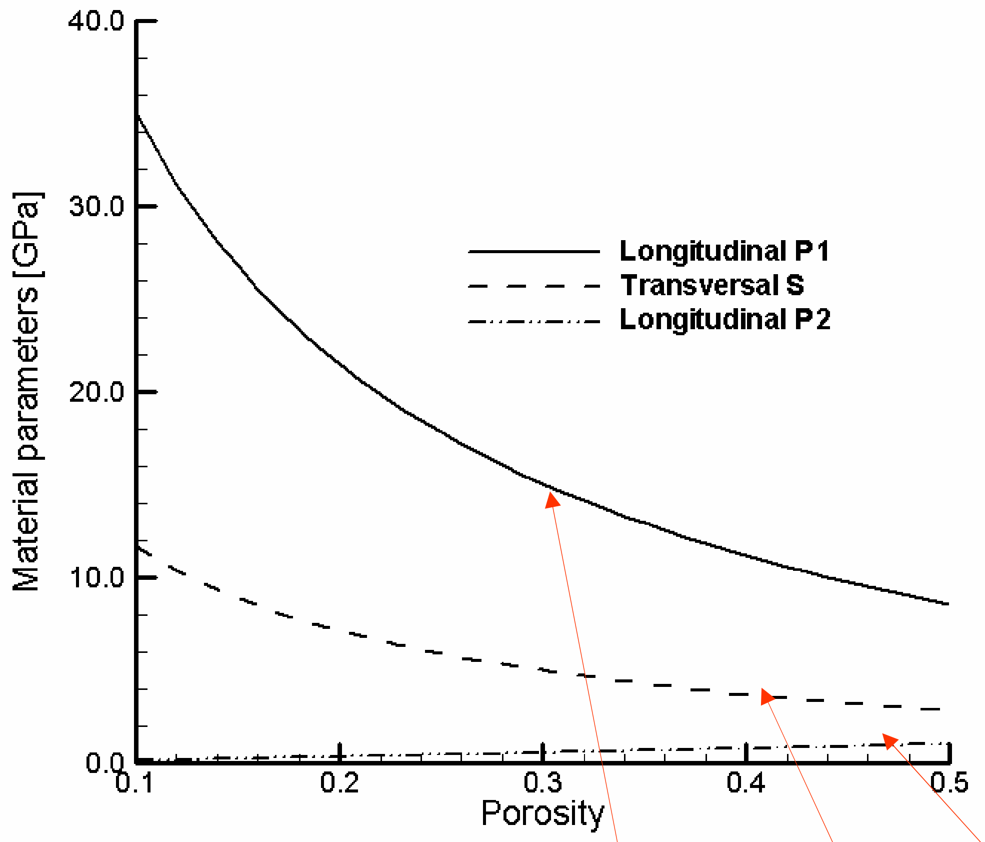
The micro-macrorelation follows provided

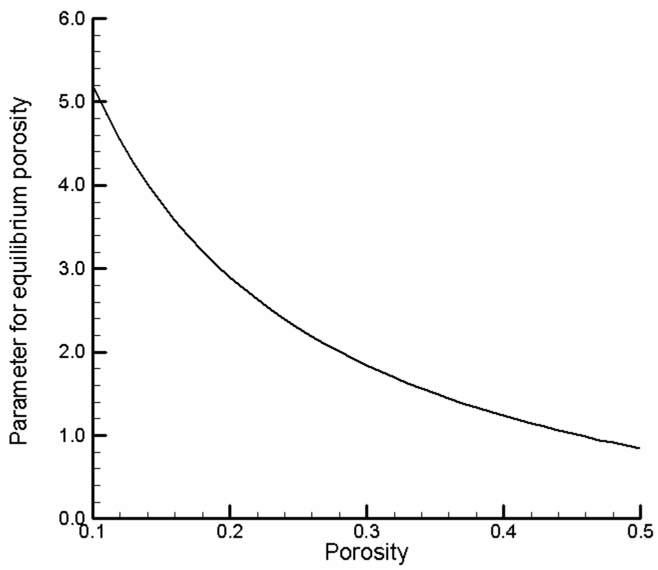
$$\tau\to\infty, \quad \gamma=\frac{\Phi}{n_0}.$$

Numerical example

$$K_s = 48 \times 10^9\, Pa, \quad K_f = 2.25 \times 10^9\, Pa$$

$$\text{Geertsma (empirical)}: \quad K_d = \frac{K_s}{1 + 50 n_0}.$$

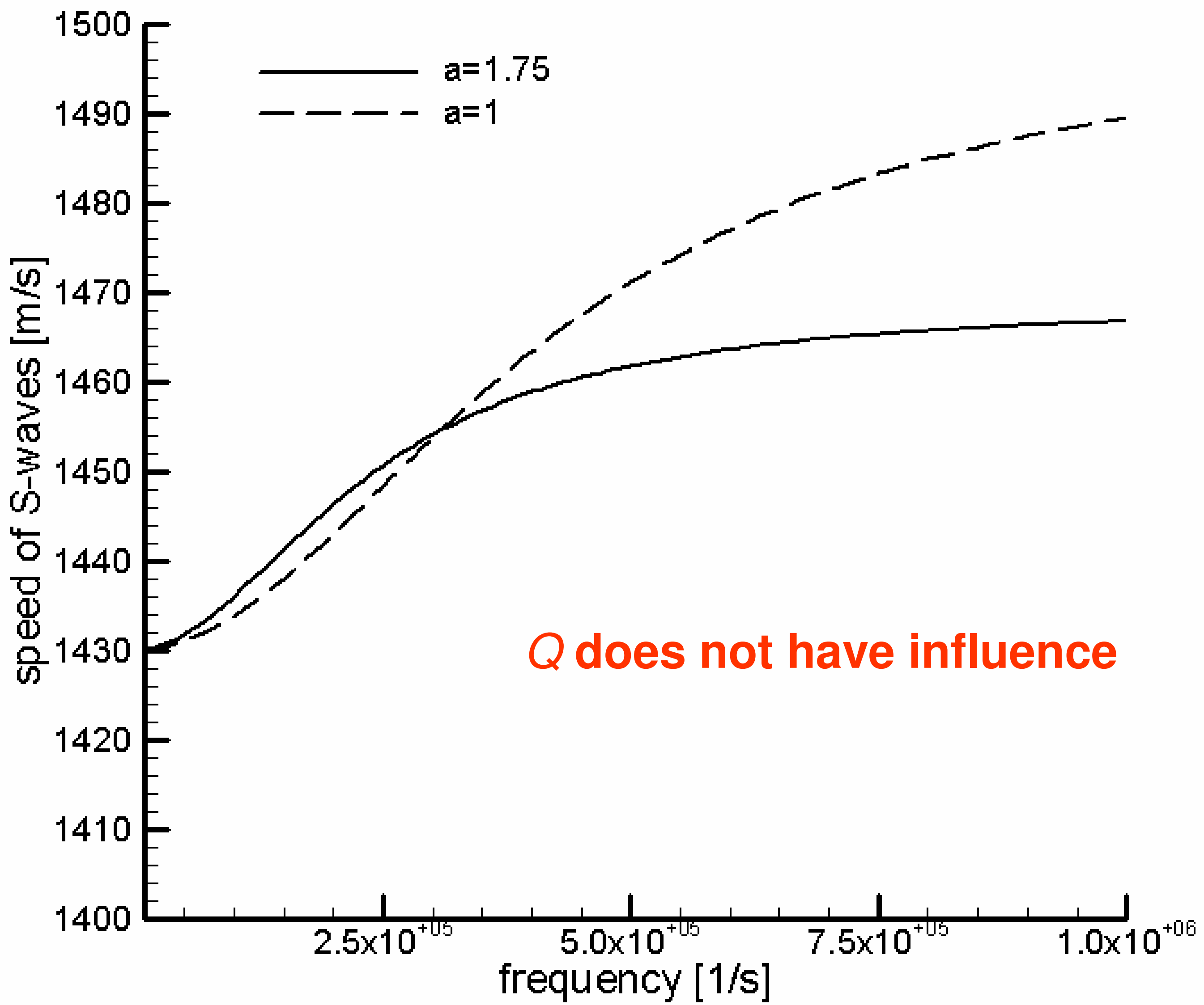




Coefficient δ for changes of porosity

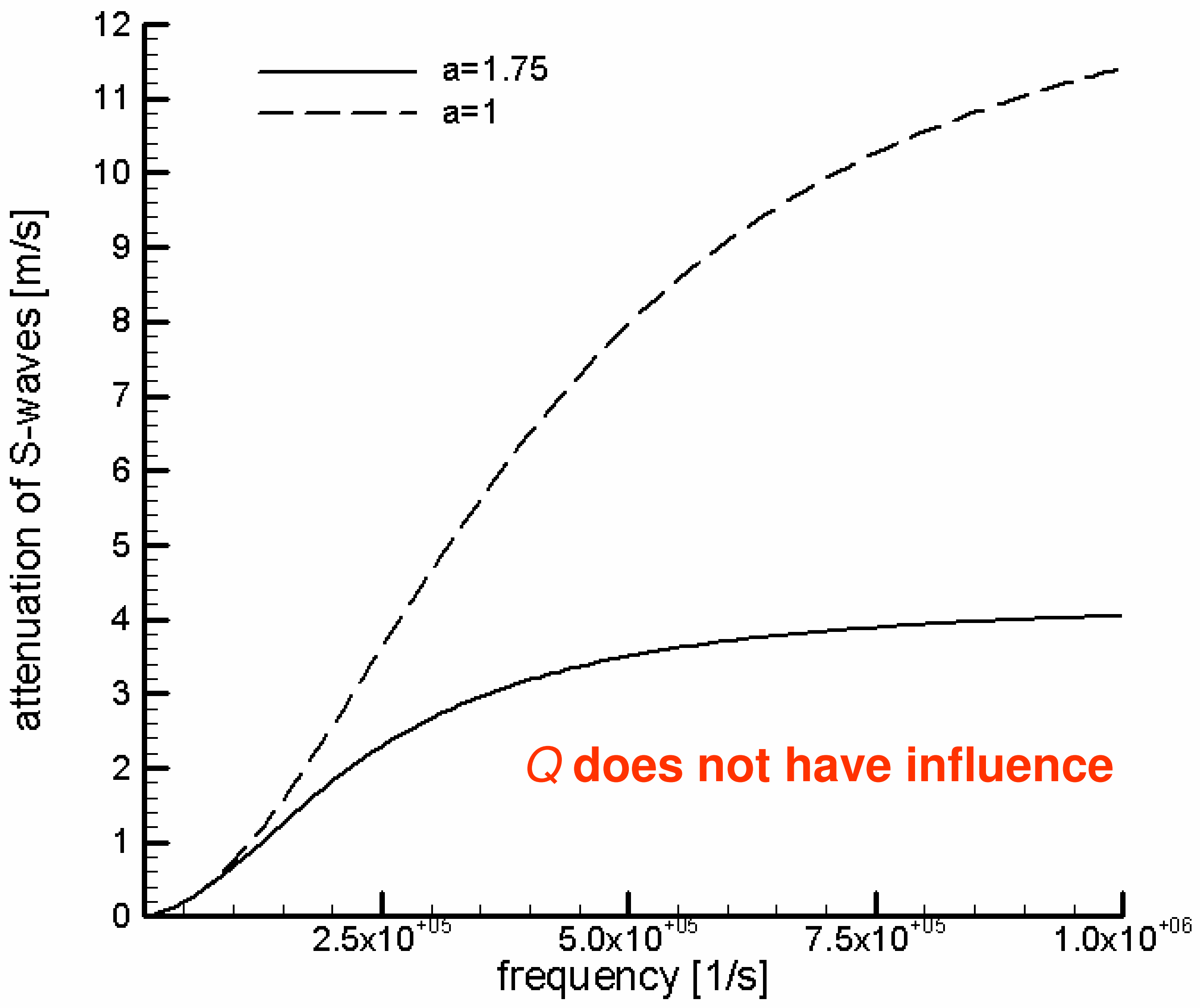
Material parameters: $\lambda^S + 2\mu^S, \quad \mu^S, \quad \rho_0^F \kappa$

Monochromatic S-wave: velocity



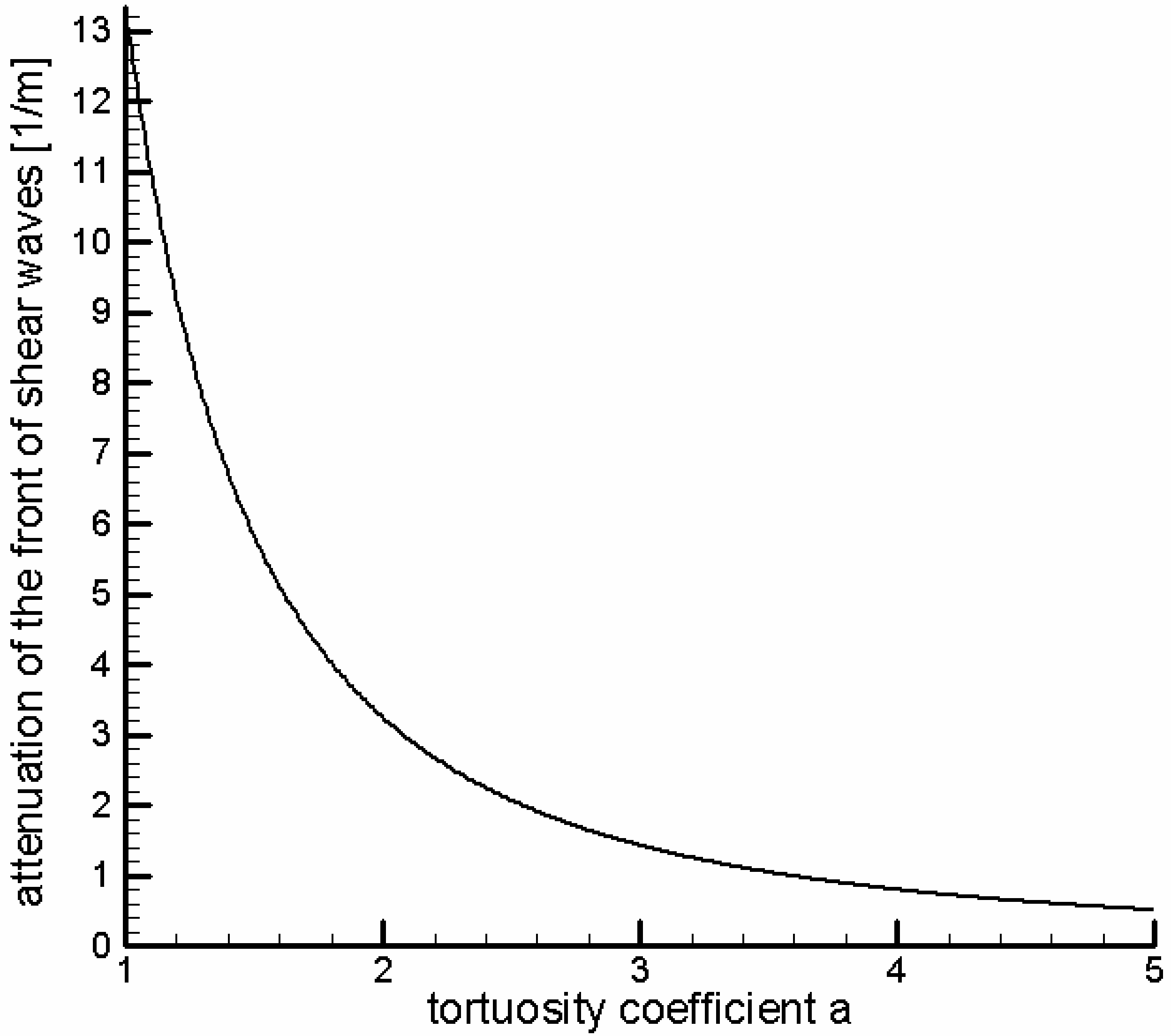
***Q* does not have influence**

Monochromatic S-wave: attenuation

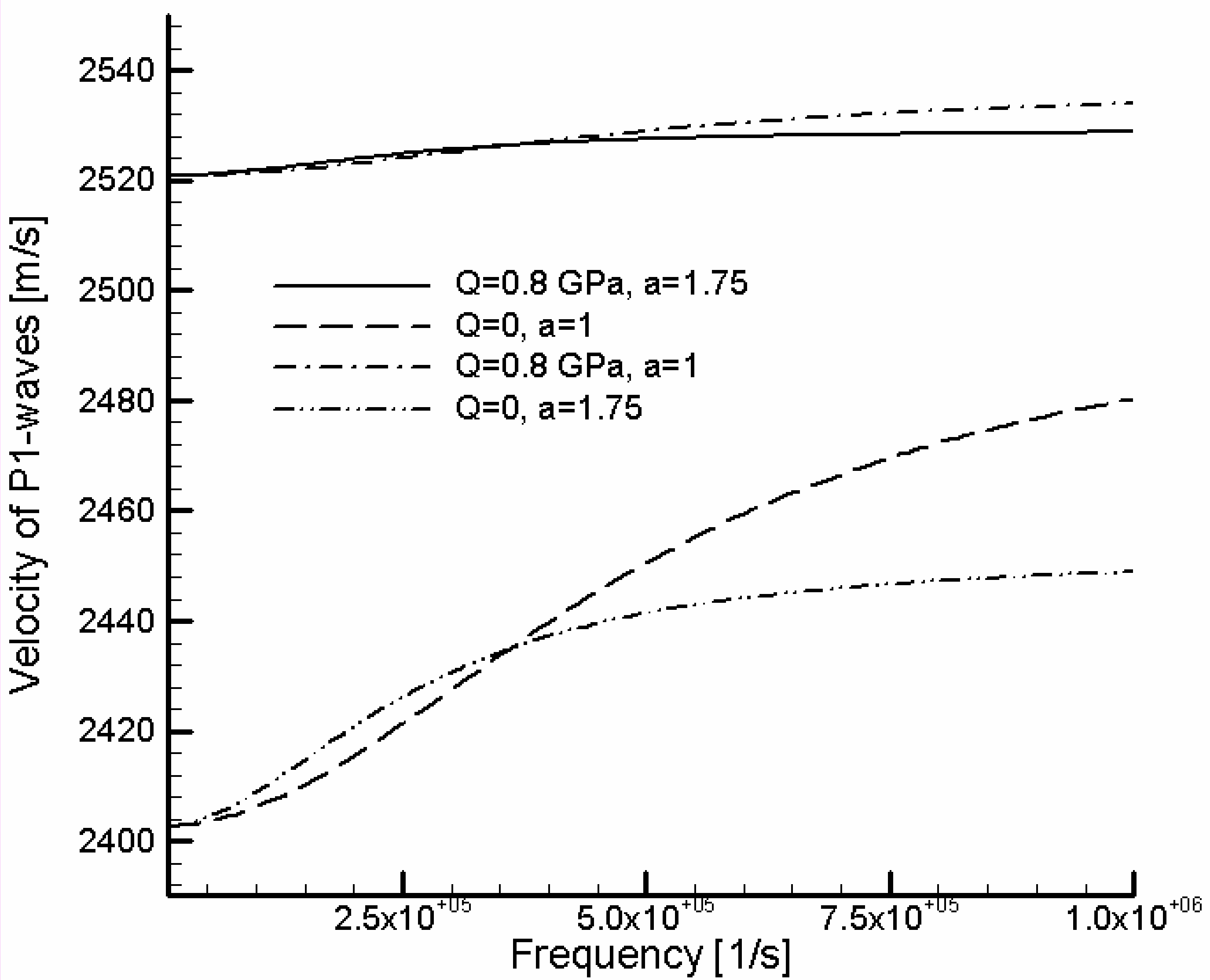


_Q_ does not have influence

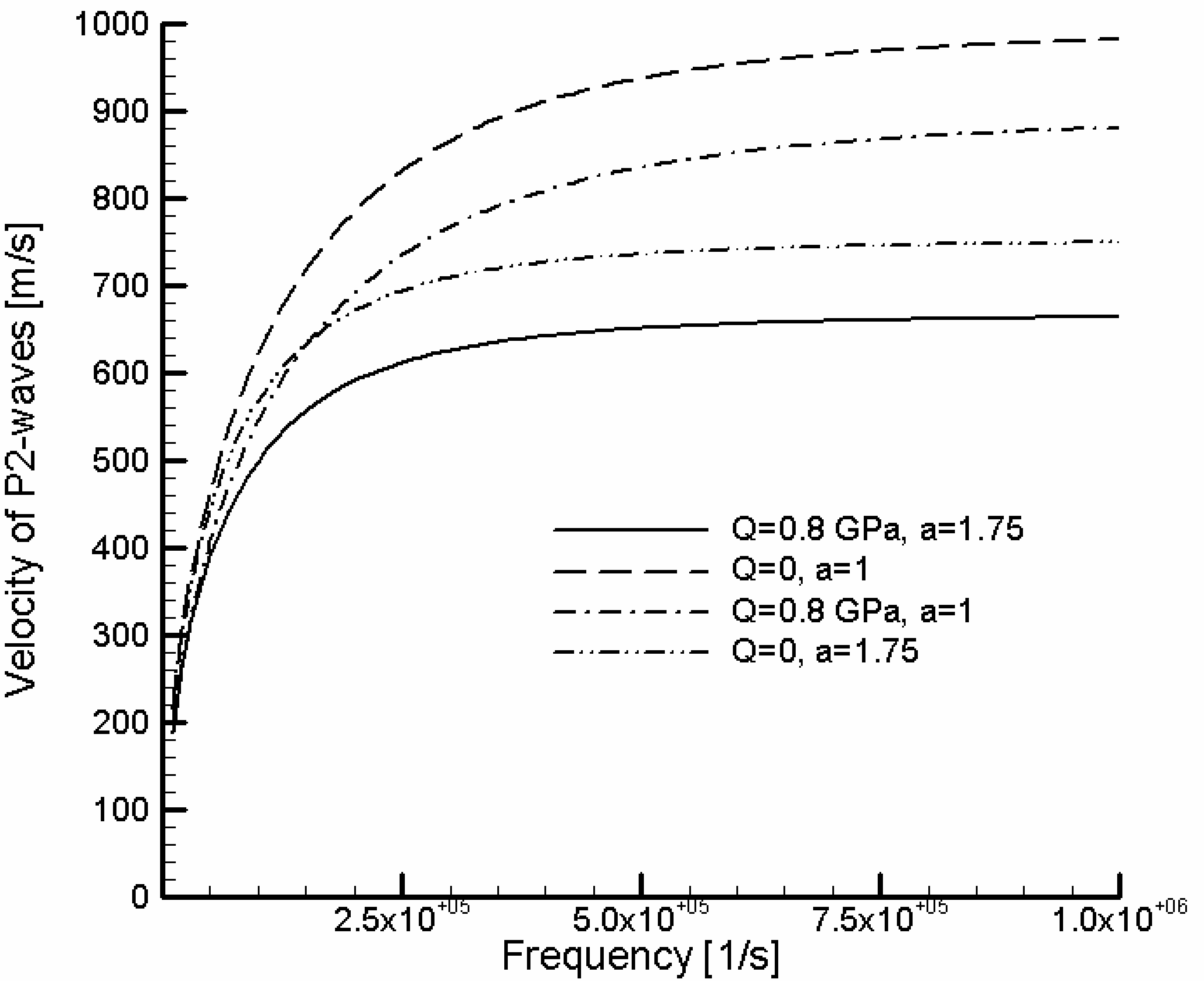
S-wave: attenuation of the front as a function of tortuosity



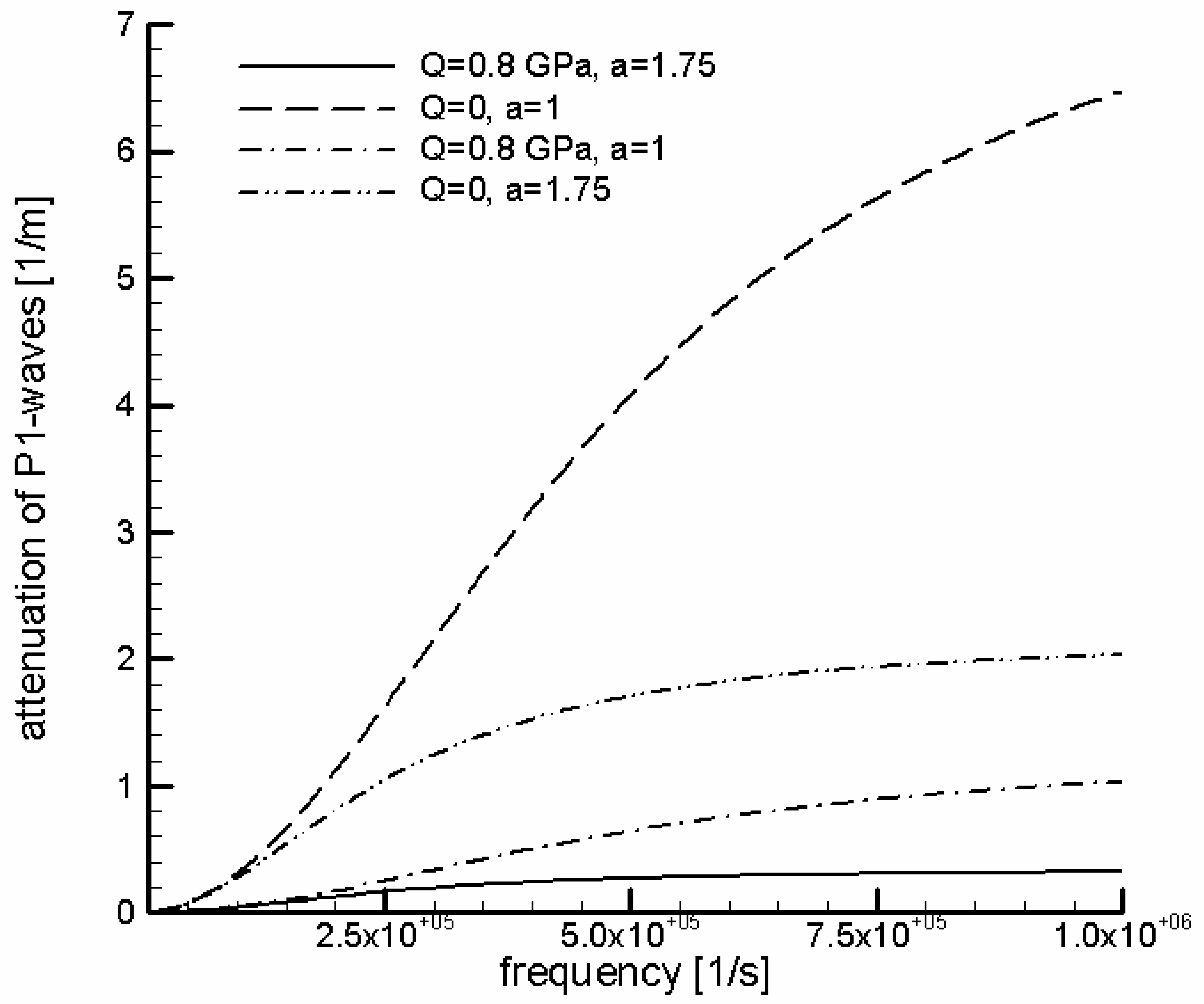
Monochromatic P1-wave: velocity



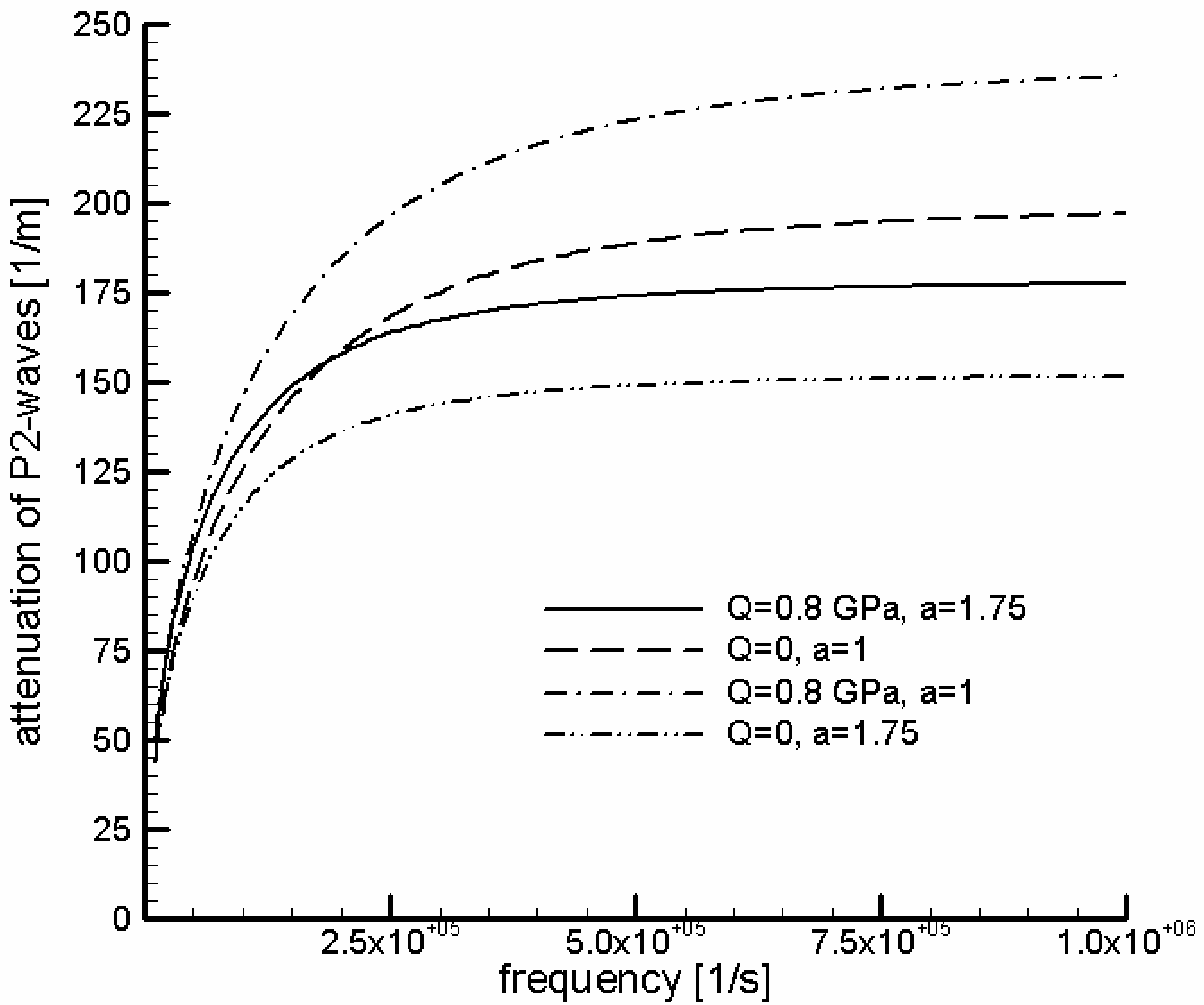
Monochromatic P2-wave: velocity



Monochromatic P1-wave: attenuation



Monochromatic P2-wave: attenuation



Conclusions:

1. **Properties of linear acoustic waves depend on Biot‘s contributions (relative accelerations and coupling of partial stresses) in a quantitative but not qualitative manner. Both models are hyperbolic (under a condition on the coupling parameter Q).**

2. **Influence of tortuosity through relative accelerations (Biot‘s model) yields the wrong behavior of the wave damping (tortuosity <u>reduces</u> the attenuation). This indicates that an influence of tortusity should enter a linear model rather through the permeabilty coefficient π which would then describe microscopic viscosity, tortuosity, scattering of waves etc. Added mass (relative acceleration) may have a bearing in the theory of suspension but not in the po–rous materials.**

3. **The Biot‘s coupling parameter Q flattens the velocity of monochromatic P1-waves contrarily to the experience. The difference between the low frequency velocity (both components move in a synchronized way) and the high frequency velocity (no influence of the fluid component) are better reflected in the simple mixture model. It indicates that Gassmann relations may give too big values of this coefficient.**

4. **Simple mixture model seems to give qualitatively reliable results and, consequently, it can be used in the analysis of such complex problems as the propagation of surface waves.**