# A few remarks on Biot's model and linear acoustics of poroelastic saturated materials

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## Abstract

This review article consists of two parts. The first part concerns the admissibility of four contributions characteristic for Biot's model of poroelastic materials: coupling of partial stresses, presence of relative accelerations in equations of motion, dependence of permeability on frequency and changes of porosity. The second part is devoted to the demonstration of those contributions in the analysis of acoustic waves. Propagation of fronts, monochromatic waves and surface waves are considered.
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## 1. Preliminaries

### 1.1. Linear models of saturated porous materials

Multicomponent continuous modeling of saturated porous materials begun some 80 years ago with the description of consolidation processes in soil mechanics. Terzaghi (e.g. [1,2]) in his works relies on the classical elasticity theory supplemented by Darcy's law for the flow of the fluid in pores. This two-component approach to consolidation has been continued in works of Biot [3],[1] the first one published in 1941, by Frenkel [5] (1944) and by Heinrich and Desoyer [6] (1955–1956). The culmination of this research were the works of Biot published in the years 1955–1956 (e.g. [7,8]). These works form until today the foundation for the linear acoustics of porous media, and their importance for the field of poroelasticity can be only compared with the role played by works of Hooke and Lamé in the classical theory of elasticity. The fundamental equations proposed by Biot can be written

in the following form:

$$\rho_0^S \frac{\partial^2 \mathbf{u}}{\partial t^2} = \operatorname{div} \mathbf{T}^S + \pi\left(\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{u}}{\partial t}\right) - \rho_{12}\left(\frac{\partial^2 \mathbf{U}}{\partial t^2} - \frac{\partial^2 \mathbf{u}}{\partial t^2}\right),$$

$$\rho_0^F \frac{\partial^2 \mathbf{U}}{\partial t^2} = -\operatorname{grad} p^F - \pi\left(\frac{\partial \mathbf{U}}{\partial t} - \frac{\partial \mathbf{u}}{\partial t}\right) + \rho_{12}\left(\frac{\partial^2 \mathbf{U}}{\partial t^2} - \frac{\partial^2 \mathbf{u}}{\partial t^2}\right),$$

$$(1)$$

where

$$\mathbf{T}^S = \mathbf{T}_0^S + (P - 2N)(\operatorname{div} \mathbf{u})\mathbf{1} + 2N\operatorname{sym} \operatorname{grad} \mathbf{u} + Q(\operatorname{div} \mathbf{U})\mathbf{1},$$

$$p^F = p_0^F - Q\operatorname{div} \mathbf{u} - R\operatorname{div} \mathbf{U}. \qquad (2)$$

In these equations $\mathbf{u}$ and $\mathbf{U}$ denote displacements of the skeleton and of the fluid, respectively. The choice of material parameters $P, N, Q, R$ describing constitutive relations for partial stresses is arbitrary. In Section 3.1, we show some other choices. Biot himself was changing his notation from one work to the other. An essential extension of the set of parameters which characterize separate components (i.e. $P, N$ for the skeleton and $R$ for the fluid) is the parameter $Q$ which introduces a coupling between stresses. The operators div, grad refer to differentiation in *Eulerian* $\mathbf{x}$-coordinates and $\operatorname{sym} \operatorname{grad} \equiv \frac{1}{2}(\operatorname{grad} + (\operatorname{grad})^T)$.

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[1]The collection of Biot's papers on porous materials has been published by Tolstoy [4].

Initial partial mass densities $\rho_0^S$ and $\rho_0^F$ were denoted in a different way by Biot. We introduce them here in order to expose the presence of the relative acceleration which appears with the material parameter $\rho_{12}$. This contribution was introduced by Biot in order to account for added mass effects which he expected to have in diffusive processes due to a complex geometry of microstructure of porous materials.

The permeability coefficient $\pi$ was also introduced in a different form by Biot. It was argued that this coefficient describing a reaction on relative motion of components should be dependent on viscosity of the fluid and, primarily, on the frequency of waves. The latter was attributed by Biot to the tortuosity.

Finally, Biot was considering increments of stresses with respect to constant initial stresses but he never mentioned this in an explicit form. For this reason, relations (2) contain initial stresses $\mathbf{T}_0^S, p_0^F$.

Numerous theoretical and experimental papers based on Biot's model prove that Biot's intuition was right and that he included in his model in a correct way the most important effects appearing in porous materials. However, the experience of the last 50 years in continuum thermodynamics gives rise to the following questions:

(1) Is the coupling of stresses described by the material parameter $Q$ admissible from the thermodynamic point of view?
(2) Is the contribution of relative accelerations admissible from the point of view of material objectivity?
(3) How are changes of porosity described by this model?
(4) How should one write in the mathematically correct form the frequency-dependent permeability?
(5) Can one extend in a consistent way Biot's model to large deformations of the skeleton and other nonlinear effects?

The first question is motivated by the experience with the theory of mixtures of ideal fluids. For such a mixture the coupling between partial pressures cannot be incorporated into the model in a thermodynamically admissible way if one does not account for a constitutive dependence on the so-called higher gradients [9]. The second law of thermodynamics yields without those gradients a model which is called the *simple mixture* in which there is no interaction term in constitutive relations for partial pressures.

One of the fundamental principles of any macroscopic continuum model is the so-called *material frame indifference* or *material objectivity* [9–12] which states that constitutive relations should be invariant with respect to the change of the observer. The relative acceleration appearing in the Biot model violates this principle [13] and yields existence of terms in equations of motion which depend simultaneously on the choice of the reference system (i.e. observer) and on the material.

Biot did not make any contributions to describe changes of porosity. There were even claims in the literature that the model does not account for such changes. The question arises if this is indeed the case.

The form of the permeability coefficient in which a dependence on a frequency of waves is incorporated cannot appear in general equations of motion which contain as well a dependence on time. Many papers on this subject avoid this problem by writing Eqs. (1) after Fourier transformation [67]. The question arises how to incorporate such a dependence in a general case when, for instance, a complex impulse is applied as a loading and the temporal form of equations is more convenient for the formulation of the problem.

In a series of recent papers I have addressed these questions. In order to find an answer a model in a fully nonlinear form had to be constructed and then linearized. As it is frequently the case with linear models, it has been shown that Biot's model indeed follows from some nonlinear extensions which satisfy both the second law of thermodynamics and the principle of material objectivity.

In this review article, these results are demonstrated without going into technical details. The new contribution concerns the admissible form of permeability which leads to the frequency dependence of this contribution in its Fourier form. Processes are assumed to be isothermal, i.e. an influence of temperature is neglected. This was an assumption made by Biot as well. It may not be particularly good in some problems of acoustics of porous materials and it is, of course, not good at all in problems of energy transfer but we want to limit attention only to Biot's model in its original isothermal form. The second part of the article dealing with acoustic waves demonstrates some basic features of linear acoustic waves in juxtaposition of the full Biot model and the simple mixture model which follows from Biot's model by neglecting the influence of the coupling of stresses and tortuosity. In this analysis a dependence of permeability on the frequency (hereditary effects) is neglected, i.e. $\pi = $ const. This may have a small influence on the quantitative behavior of speeds of propagation and a bigger influence on the quantitative behavior of attenuations. However, no qualitative differences are expected. Finally, some remarks on boundary conditions and surface waves are subject of the last part. In this part only the simple mixture model is considered.

Unfortunately, the list of references contains only a few representative works. There exist a few thousand publications on Biot's model and it is even difficult to point out the priorities.

### 1.2. Macroscopic fields and their relation to microstructure

Description of porous materials based on the theory of mixtures is purely macroscopic and its fields do not have to be related to any true quantities characterizing components of the mixture. In such a heuristic approach one can rely on experimental observations of the mixture as a whole, and no micro–macro transition procedure is needed in order to solve problems. However, in the case of porous materials

such a separation is not practicable. Data for true materials are easily available, and they are used in the construction of macroscopic field models. This yields, of course, advantages but also some confusions. The typical example are mass densities which appear in the form of partial mass densities of the mixture as well as true mass densities of real components. The same concerns the partial pressure of the fluid component and the so-called pore pressure. It is often the case that these two sorts of quantities are so mixed together that confusions arise.

In this work we try to keep these two classes of quantities separated from each other. The formal description of the system is based on the continuum theory of mixtures and only fields appropriate for such a description appear in the field equations. Relations to true quantities are discussed separately and are not part of the mathematical model.

The model of porous saturated materials is based on the two-component mixture theory. In a fully nonlinear theory formulated on a reference configuration (Lagrangian description) the isothermal model must contain the following fields:

1. Partial mass densities $\rho^S, \rho^F$ which are functions of the spacial variable $\mathbf{X}$ and time $t$; integrated over a finite domain they give the mass of a particular component (the skeleton or the fluid) contained in this domain.
2. The function of motion of the skeleton (of the solid frame) $\mathbf{f}^S$ which specifies the position of a chosen point $\mathbf{X}$ at any instant of time $t$. Its derivatives with respect to $\mathbf{X}$ and $t$ define the deformation and the velocity of the skeleton, respectively.
3. The velocity of the fluid $\mathbf{v}^F$ which specifies the velocity of the fluid particle which instantaneously occupies the same position as the point $\mathbf{X}$ of the solid frame.
4. The porosity $n$ which describes the fraction of voids of the solid frame. This is a microstructural variable characteristic for the theory of *immiscible mixtures*.

We present some details of this Lagrangian description in Section 2.1. but in linear theories the set of fields is chosen in a different way well known from the classical linear elasticity. They are formulated on the current configuration (Eulerian description) and, due to linearity, this description is identical with Lagrangian description (for details, e.g. see: [11]). The fields are as follows:

1. Current mass densities $\rho_t^S$, $\rho_t^F$. The subscript $t$ indicating the reference to the current location is usually dropped as in the linear model there is no distinction between Eulerian and Lagrangian description.
2. The displacement of the skeleton $\mathbf{u}^S$. Its derivative with respect to $\mathbf{x}$ defines the deformation tensor. In the linear theory this is the Almansi–Hamel deformation tensor, $\mathbf{e}^S$. The time derivative gives the velocity of the skeleton $\mathbf{v}^S$. Volume changes are specified by the first invariant of the Almansi–Hamel tensor, $e = \text{tr}\,\mathbf{e}^S$. If there is no mass exchange between components this, in turn, determines

uniquely changes of the partial mass density and the field of mass density $\rho_t^S$ becomes superfluous.
3. The velocity of the fluid $\mathbf{v}^F$. It is customary in Biot's model to work with the displacement of the fluid $\mathbf{U}$. However, as in fluid mechanics in general, this quantity does not have any physical meaning and it is not sought in solving boundary value problems. It plays only an auxiliary role in the model and defines the velocity of the fluid $\mathbf{v}^F = \partial \mathbf{U}/\partial t$ as well as its volume changes $\varepsilon = \text{div}\,\mathbf{U}$. As in the case of the skeleton, through the partial mass balance equation the latter determines uniquely changes of the partial mass density, $\rho_t^F$, if there is no mass exchange between components.
4. The porosity $n$ (denoted by $\phi$ in many works on Biot's model). In Biot's model this quantity does not appear in explicit form. However, its changes can be calculated in terms of volume changes of both components and material parameters (compressibilities) appearing in the model. Hence, even though not constant, the porosity is not listed among the fields of Biot's model.

The above list does not contain any "true" quantities such as true mass densities of components or pore pressure because they do not belong to the macroscopic description. However, on mass densities we illustrate reasons for not choosing these quantities as fields. For porous materials a common way to transfer quantities from the "micro"-level of true substances to the "macro"-level of the mixture (continuum) theory is to perform averaging over the so-called *representative elementary volume* (REV) which is large in comparison with microstructural dimensions (e.g. radius of channels or true stream lines) and small in comparison with macroscopic dimensions. Fields on the "micro"-level are localized in their own components, i.e. they are defined only on subdomains in which a particular component is instantaneously located. If we denote by $\rho_{\text{micro}}^{SR}, \rho_{\text{micro}}^{FR}$ the true (real) mass densities of the solid phase and the fluid phase then averaging over REV has the following form:

$$\rho^S(\mathbf{x}, t) = \frac{1}{V} \int_{\text{REV}(\mathbf{x},t)} \rho_{\text{micro}}^{SR}(\mathbf{z}, t) H^S(\mathbf{z}, t)\, \mathrm{d}V,$$

$$\rho^F(\mathbf{x}, t) = \frac{1}{V} \int_{\text{REV}(\mathbf{x},t)} \rho_{\text{micro}}^{FR}(\mathbf{z}, t) H^F(\mathbf{z}, t)\, \mathrm{d}V, \tag{3}$$

where $\text{REV}(\mathbf{x}, t)$ denotes the domain of the representative elementary volume located at point $\mathbf{x}$ at instant of time $t$, $V$ is its volume, usually assumed to be constant, $H^S, H^F$ are the characteristic functions of the domains of the skeleton and of the fluid, respectively. This means that $H^S(\mathbf{z}, t)$ is equal to 1 when the point $\mathbf{z}$ is occupied at the instant of time $t$ by a particle of the skeleton and zero otherwise. $H^F$ has the same structure for fluid particles and it means that $H^S + H^F = 1$ everywhere. The quantities may substantially fluctuate within the representative elementary volume. If this is not the case and we can assume approximately that they are constant within $\text{REV}(\mathbf{x}, t)$ and possess values, say,

$\rho^{SR}(\mathbf{x}, t), \rho^{FR}(\mathbf{x}, t)$ then we have

$$\rho^S \approx (1-n)\rho^{SR}, \quad \rho^F \approx n\rho^{FR},$$

$$n := \frac{1}{V} \int_{REV(\mathbf{x},t)} H^F(\mathbf{z}, t) \, dV, \tag{4}$$

where $n$ is the porosity which is the fraction of the volume of voids to the full volume of REV. Clearly, the true mass densities $\rho^{SR}$, $\rho^{FR}$ may approximately and locally coincide with partial mass densities of the real skeleton and of the real fluid, but they do not possess the property that multiplied by a macroscopic volume they give the mass of the corresponding component within this volume. The results must be corrected by taking the product with the corresponding volume fraction: $n$ for the fluid and $1-n$ for the skeleton. This is indeed being done in the works on Biot's model in which these true mass densities are used. They are then usually denoted $\rho_s \equiv \rho^{SR}$, $\rho_f \equiv \rho^{FR}$.

On the other hand, the notion of the *pore pressure p* seems to be physically more important than its partial counterpart—the partial pressure $p^F$—which is one third of the negative trace of the partial stress tensor in the fluid. It appears, for instance, as an important part in the definition of *effective stresses* in theories of plasticity of soils, it contributes to the definition of material parameters for granular materials depending on the so-called *confining pressure*, etc. Consequently, the transition from the microlevel to the macrolevel is in this case much more essential. Unfortunately, it is also much more complicated. In an arbitrary cross-section of a porous material with the orientation $\mathbf{n}$ (a unit vector perpendicular to the surface of the cross-section) the resultant force acting on a macroscopic small surface $dA$ carried by the inviscid fluid in the normal direction is $p^F \mathbf{n} \, dA$. The true fluid is carrying on this surface the force $p\mathbf{n} \, dA^F$, where $dA^F$ denotes the part of $dA$ on which the cross-section cuts the voids. The equilibrium condition yields

$$\frac{p}{p^F} = \frac{dA}{dA^F}. \tag{5}$$

According to the Delesse Theorem [14] the fraction of surfaces is identical with the volume fraction $n$ if the microstructure is random. Consequently, for such porous materials

$$p^F = np. \tag{6}$$

This relates the macroscopic partial pressure $p^F$ and the pore pressure $p$. The above condition may hold in the bulk but it is rather doubtful that it also holds near boundaries of porous materials where as a rule there exists a boundary layer in which one has to account for different flow conditions. This is one of the reasons that volume averaging indicated above predicts a structure of balance equations for two-component porous materials but it does not yield macroscopic constitutive relations. This problem is presented in the article of Gray in [15, pp. 67–130]. It is only exceptional that such relations can be found for, e.g.

compressibilities. They are given by Gassmann relations for Biot's model and we discuss them further in this article.

### 1.3. Thermodynamics, simple mixtures, dissipation

Recent developments of continuum thermodynamics (e.g. [9,11,16,17]) allow to construct a fully nonlinear theory of immiscible mixtures. The strategy of thermodynamic modeling relies on the second law of thermodynamics. After construction of general field equations one imposes the condition that the entropy inequality

$$\frac{\partial \rho \eta}{\partial t} + \text{Div}\,\mathbf{H} \geqslant 0 \tag{7}$$

should be satisfied for all solutions of field equations. In this inequality $\eta$ is the entropy density per unit mass of the mixture in a reference configuration and $\mathbf{H}$ is the entropy flux. Both these fields are assumed to depend on the same constitutive variables as all other constitutive quantities of the model. The choice of constitutive variables defines the specific model. For instance, for poroelastic materials constitutive variables should contain at least the following variables:

1. partial mass densities of components, $\rho^S, \rho^F$,
2. deformation of the skeleton, $\mathbf{e}^S$,
3. relative velocity, $\mathbf{v}^F - \mathbf{v}^S$. This combination is necessary with respect to the principle of material objectivity,
4. porosity, $n$,
5. temperature, $T$,
6. temperature gradient, Grad $T$.

In the above relations, operators Grad and Div refer to differentiation in *Lagrangian* $\mathbf{X}$-coordinates.

The last two variables do not appear if we assume that processes are isothermal.

As consequence of the second law of thermodynamics we obtain two classes of relations: (i) thermodynamic identities and (ii) the residual inequality which specifies the dissipation in the system. It is remarkable that for nondissipative systems thermodynamic identities coincide with results of the variational principle. For instance, one obtains the result that partial stresses are specified by derivatives of a thermodynamic potential which replaces the Lagrangian density of the action functional in the variational formulation. However, in contrast to the variational formulation thermodynamic identities hold true also in nonequilibrium when the dissipation is unequal to zero.

For poroelastic materials numerous nonlinear thermodynamic models were investigated (e.g. [18–21]). The common conclusion of these investigations is that the above choice of variables is not sufficient to produce Biot's model of poroelastic materials. The second law of thermodynamics eliminates from the model a coupling between partial stresses and yields the so-called *simple mixture model*. Such models are known for fluid mixtures [9]. For a simple mixture of ideal fluids, for instance, the

partial pressure of a chosen component depends on its own volume changes but does not react to volume changes of other components. In order to account for such couplings one has to introduce a constitutive dependence on higher gradients of fields. Physically they reflect the reaction of a particular component to changes of states of other components in a small spatial vicinity of a chosen particle (weak nonlocality). For mixtures of fluids it means that gradients of partial mass densities must be constitutive variables. The same requirement can be applied to poroelastic materials. A dependence on the gradient of porosity is already sufficient to produce the coupling between partial stresses proposed by Biot [19]. We demonstrate this result for a linear model.

The dissipation which follows from the entropy inequality gives hints on the modeling of the evolution of those quantities which yield the dissipation and, consequently, a relaxation of the system to the thermodynamic equilibrium. For isothermal processes in poroelastic materials the dissipation inequality has a very simple form

$$\mathscr{D} := \hat{\mathbf{p}} \cdot (\mathbf{v}^F - \mathbf{v}^S) \geqslant 0, \tag{8}$$

where $\hat{\mathbf{p}}$ denotes the momentum source (the diffusive force) in the partial momentum balance equations. In Section 3.3 we present the details of this concept. Obviously, the dissipation $\mathscr{D}$ is zero if the velocities of components are equal: $\mathbf{v}^F = \mathbf{v}^S$. Processes in which this condition holds are called *equilibrium processes*. In equilibrium processes there is no relative motion—diffusion—of components. Simultaneously, inequality (8) demonstrates that equilibrium processes minimize the dissipation. Hence, the matrix of second derivatives of $\mathscr{D}$ with respect to relative velocities (Hessian) must form a positive definite matrix. In the linear model $\hat{\mathbf{p}}$ is linear in relative velocities and the material parameters entering the definition of $\hat{\mathbf{p}}$ must fulfil this restriction. In the simplest case it means that the diffusion coefficient has to be positive.

Let us mention in passing that the structure of the momentum source, $\hat{\mathbf{p}}$, determines the form of Darcy's law in the simplified model in which we neglect the inertial term in the momentum balance for the fluid. Linearity of $\hat{\mathbf{p}}$ with respect to the relative velocity $\mathbf{v}^F - \mathbf{v}^S$ may be acceptable as a first approximation in the theory of acoustic waves but, in general, it is not well justified by experiments. There exist various nonlinear corrections (e.g. see [22] for a critical review and some possible corrections) but we shall not enter this problem as it goes beyond the Biot model.

## 2. Governing equations

### 2.1. Mass balance, volume changes

As indicated in the previous section two-component mixture models deal with macroscopic fields defined in almost every point of the system, i.e. in all points except a set whose volume is zero. Consequently, one cannot speak about such quantities as volume of the fluid or volume of the skeleton, true mass densities, etc. These notions do not belong to the macroscopic model. This means that, in particular, the motion of the skeleton (the solid frame of the porous material) is described by the function

$$\mathbf{x} = \mathbf{f}^S(\mathbf{X}, t), \quad \mathbf{X} \in \mathscr{B}_0 \subset \mathfrak{R}^3, \tag{9}$$

which assigns a position $\mathbf{x}$ to a material point $\mathbf{X}$ at any instant of time $t$. This function describes the *trajectory* of the point $\mathbf{X}$. Its *deformation gradient* $\mathbf{F}^S = \mathrm{Grad}\,\mathbf{f}^S$ describes changes (rotations and elongations) of vectors tangent to arbitrary material curves from the reference set $\mathscr{B}_0$ and the time derivative $\mathbf{v}^S = \partial\mathbf{f}^S/\partial t$ is the *velocity* of the point $\mathbf{X}$. Arbitrary large deformations of the skeleton can be, for instance, described by the following *Almansi–Hamel deformation tensor*

$$\mathbf{e}^S = \tfrac{1}{2}(\mathbf{1} - \mathbf{F}^{-T}\mathbf{F}^{-1}), \tag{10}$$

where T denotes the transpose of the matrix. For small deformations, when the following conditions are satisfied:

$$\|\mathbf{e}^S\| \ll 1, \quad \|\mathbf{e}^S\| := \max\{|\lambda^{(1)}|, |\lambda^{(2)}|, |\lambda^{(3)}|\},$$

$$\det(\mathbf{e}^S - \lambda\mathbf{1}) = 0, \tag{11}$$

i.e. $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ are eigenvalues of $\mathbf{e}^S$ (principal stretches), there is no distinction between Lagrangian variables $\mathbf{X} \in \mathscr{B}_0$ and Eulerian variables $\mathbf{x} \in \mathbf{f}^S(\mathscr{B}_0, t)$ and one can introduce the *displacement* of the skeleton

$$\mathbf{f}^S = \mathbf{X} + \mathbf{u}^S, \quad \mathbf{u}^S = \mathbf{u}^S(\mathbf{x}, t) \Longrightarrow \mathbf{e}^S = \mathrm{sym}\,\mathrm{grad}\,\mathbf{u}^S,$$

$$\mathbf{v}^S = \frac{\partial\mathbf{u}^S}{\partial t}, \tag{12}$$

where 'grad' means the differentiation with respect to the Eulerian variable $\mathbf{x}$.

Volume changes are described by the Jacobian $J^S$ of the deformation gradient $\mathbf{F}^S$ as for a usual change of variables of integration. For instance, the mass of the skeleton of an arbitrary portion $\mathscr{P}_0 \subset \mathscr{B}_0$ changes due to the motion in the following way:

$$M^S(\mathscr{P}_0) = \int_{\mathscr{P}_0} \rho_0^S \,\mathrm{d}V_0 = \int_{\mathbf{f}^S(\mathscr{P}_0, t)} \rho^S \,\mathrm{d}V, \quad \mathrm{d}V = J^S \,\mathrm{d}V_0$$

$$\Longrightarrow \rho^S = \rho_0^S J^{S-1}, \quad J^S := \det \mathbf{F}^S > 0. \tag{13}$$

The above formula is the solution of the partial mass balance equation for the skeleton for systems without mass exchange between components. For volume changes under small deformation we can easily relate the Jacobian $J^S$ to the first invariant of the Almansi–Hamel deformation tensor. In the general case

$$(J^S)^2 = \det(\mathbf{F}^S\mathbf{F}^{ST}). \tag{14}$$

Simultaneously, the eigenvalue problem for the symmetric tensor $\mathbf{F}^S\mathbf{F}^{ST}$:

$$(\mathbf{F}^S\mathbf{F}^{ST} - \beta\mathbf{1})\mathbf{k} = 0 \Longrightarrow \lambda = \frac{1}{2}\left(1 - \frac{1}{\beta}\right), \tag{15}$$

yields

$$(J^S)^2 = \beta^{(1)}\beta^{(2)}\beta^{(3)} = \frac{1}{1 - 2\lambda^{(1)}} \frac{1}{1 - 2\lambda^{(2)}} \frac{1}{1 - 2\lambda^{(3)}}$$
$$\approx \left(1 + \sum_{\alpha=1}^{3} \lambda^{(\alpha)}\right)^2. \tag{16}$$

We have used definition (10). Hence

$$J^S \approx 1 + e, \quad e := \operatorname{tr} \mathbf{e}^S \equiv \sum_{\alpha=1}^{3} \lambda^{(\alpha)}, \tag{17}$$

i.e. volume changes of the skeleton are indeed described by the first invariant $e$ of $\mathbf{e}^S$. This yields immediately the following relation for changes of the partial mass density $\rho^S$ (compare (13)):

$$\frac{\rho_0^S - \rho^S}{\rho_0^S} = \frac{J^S - 1}{J^S} \approx e. \tag{18}$$

Simultaneously, relations (12) yield the following identity:

$$\frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad} \mathbf{v}^S \quad \text{i.e.} \quad \frac{\partial e}{\partial t} = \operatorname{div} \mathbf{v}^S. \tag{19}$$

Therefore, the mass balance equation for the skeleton is identically satisfied

$$\frac{\partial \rho^S}{\partial t} + \operatorname{div}(\rho^S \mathbf{v}^S) \approx \frac{\partial \rho^S}{\partial t} + \rho_0^S \operatorname{div} \mathbf{v}^S$$
$$= -\rho_0^S \frac{\partial e}{\partial t} + \rho_0^S \operatorname{div} \mathbf{v}^S \equiv 0. \tag{20}$$

We proceed to the description of the motion of the fluid component. In contrast to the skeleton this motion is not described by the displacement but, as customary in fluid mechanics, by the velocity field $\mathbf{v}^F(\mathbf{x}, t)$. Principally, one can integrate the velocity field to obtain a displacement field of the fluid[2] but this is usually not done because the result depends on the choice of the reference configuration and, of course, the fluid does not possess any privileged reference. Consequently, one relies on the Eulerian description of the fluid, and this requires knowledge of both the velocity $\mathbf{v}^F$ as well as the partial mass density $\rho^F$. The latter is determined by the mass balance equation

$$\frac{\partial \rho^F}{\partial t} + \operatorname{div}(\rho^F \mathbf{v}^F) = 0. \tag{21}$$

In the linear theory we assume that changes of the mass density with respect to the reference value $\rho_0^F$ appearing in the same instant of time in which the skeleton occupies the configuration $\mathscr{B}_0$ are small, i.e.

$$\left| \frac{\rho^F - \rho_0^F}{\rho_0^F} \right| \ll 1. \tag{22}$$

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[2] i.e. for a given field of velocity $\mathbf{v}^F$ we must integrate the equation

$$\frac{\partial \mathbf{x}}{\partial t} = \mathbf{v}^F(\mathbf{x}, t), \quad \mathbf{x}(t = 0) = \mathbf{X},$$

which is a set of three ordinary nonlinear differential equations for the trajectory of the fluid particle $\mathbf{X}$.

In such cases the mass balance equation (21) can be written in the form

$$\frac{\partial \varepsilon}{\partial t} = \operatorname{div} \mathbf{v}^F, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F}. \tag{23}$$

Obviously, the field $\varepsilon$ describes small volume changes of the fluid in the same way as $e$ describes small volume changes of the skeleton.

We conclude that partial volume changes of the skeleton $e$ are determined when the deformation of the skeleton $\mathbf{e}^S$ is known, while the volume changes of the fluid $\varepsilon$ follow from Eq. (23) if the velocity of the fluid $\mathbf{v}^F$ is determined by its own equation. We see further that this is the partial momentum balance equation. Volume changes of the fluid determine changes of the partial mass density of the fluid $\rho^F$.

In spite of the above objections many researchers follow M. Biot and introduce the field of displacement for the fluid component. The commonly used notation is as follows: The displacement of the skeleton is denoted by $\mathbf{u} \equiv \mathbf{u}^S$ and the displacement of the fluid $\mathbf{U}$ is introduced in such a way that $\mathbf{v}^F = \partial \mathbf{U}/\partial t$. Then time integration in (19) and (23) yields the following relations for volume changes:

$$e = \operatorname{div} \mathbf{u}, \quad \varepsilon = \operatorname{div} \mathbf{U}. \tag{24}$$

In some of his works Biot uses also the *increment of fluid contents* as a field replacing volume changes of the fluid. This field is defined in the following way:

$$\zeta = n_0(e - \varepsilon) \equiv n_0 \operatorname{div}(\mathbf{u} - \mathbf{U}). \tag{25}$$

The latter relation yields an important conclusion

$$\frac{\partial \zeta}{\partial t} = n_0 \operatorname{div}(\mathbf{v}^S - \mathbf{v}^F). \tag{26}$$

Hence the field $\zeta$ cannot change without diffusion: there must be a relative motion of components in order to change the fluid contents. However, this means as well that changes of the fluid contents cannot appear in thermodynamic equilibrium processes. This conclusion has the paramount consequences for the applicability of variational methods in the theory of poroelastic materials. We return to this problem in the sequel.

### 2.2. Momentum balance

The form of the momentum balance equations depends on the choice of the reference system. In this note we skip the problem of reference to noninertial systems as this does not differ from the same problem in one-component systems (e.g. see [12]). Additional terms related to the centrifugal, Coriolis, Euler and translational accelerations have the similar form but refer to a corresponding kinematics of the component. In an inertial reference system these partial balance equations have the following

linearized forms:

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \operatorname{div} \mathbf{T}^S + \rho_0^S \mathbf{b}^S + \hat{\mathbf{p}},$$

$$\rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = -\operatorname{grad} p^F + \rho_0^F \mathbf{b}^F - \hat{\mathbf{p}}, \qquad (27)$$

where the current mass densities are replaced by their reference values, convective parts of momentum fluxes (i.e. $\operatorname{div}(\rho^S \mathbf{v}^S \otimes \mathbf{v}^S)$ and $\operatorname{div}(\rho^F \mathbf{v}^F \otimes \mathbf{v}^F)$, respectively) are neglected and it is assumed that the partial stress tensor in the fluid reduces to the partial pressure $p^F$. The first two simplifications follow from the linearization, while the last one means that we assume the fluid component to be ideal on the macroscopic level of description. This need not be so for the true fluid whose viscosity appears in the microstructural relation for the momentum source, $\hat{\mathbf{p}}$. We shall return further to this relation.

The quantities $\rho_0^S \mathbf{b}^S, \rho_0^F \mathbf{b}^F$ are partial external body forces per unit volume.

Frequently the partial description of momenta is replaced by the bulk momentum of the whole mixture and only one of the partial momenta. It is usually the partial momentum balance for the fluid because in some cases of practical bearing the inertial force of the fluid can be neglected; this momentum balance yields then a generalization of Darcy's law. The bulk momentum balance which is the sum of the partial equations has the form

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = \operatorname{div} \mathbf{T} + \rho_0 \mathbf{b}, \qquad (28)$$

where

$$\mathbf{T} \approx \mathbf{T}^S - p^F \mathbf{1}, \quad \rho_0 = \rho_0^S + \rho_0^F,$$

$$\rho_0 \mathbf{v} = \rho_0^S \mathbf{v}^S + \rho_0^F \mathbf{v}^F, \quad \rho_0 \mathbf{b} = \rho_0^S \mathbf{b}^S + \rho_0^F \mathbf{b}^F. \qquad (29)$$

Obviously, $\mathbf{T}$ denotes the bulk stress tensor and $\mathbf{v}$ is the so-called barycentric velocity. The latter is not a very useful field because it does not have any global physical interpretation: Indeed, diffusion separates particles of components after a finite time so much that one cannot define any reasonable collection of centers of gravity. However, in some quasistatic problems the bulk momentum balance equation may be very useful. It appears, for instance, in the theory of plastic deformations of soils in which the bulk stress tensor plays an important role.

The above structure of the momentum balance shows that the partial stress tensor in the skeleton cannot be usually identified with the so-called *effective stresses* introduced into soil mechanics by Terzaghi. As already mentioned, the partial pressure in the fluid $p^F$ is in many cases related to the so-called pore pressure $p$ by the relation $p^F = np$, where $n$ is the current porosity.

### 2.3. Changes of porosity

Porosity is the microstructural variable of the two-component immiscible mixture and it requires a relation which would determine its changes. However, we do not have any additional natural macroscopic conditions which would lead to such a relation. It must be determined either from the analysis of the microstructure or it must be introduced to the macroscopic model in a heuristic way. In the literature there appear at least five ways to describe changes of porosity:

(1) *Constitutive assumptions*; this is done in particular for porosity changes in the thermodynamic equilibrium, e.g. assuming that the equilibrium porosity is a function of volume changes of the skeleton (solid frame). In some cases it is not sufficient. In granular materials and, in particular, in nonlinear models (large deformations) there may be an influence of shearing deformations on changing porosity.

(2) *Assumption on incompressibility of true components* (e.g. [23–25]). Then partial mass balance equations yield a differential equation for porosity and a differential constraint on relative volume changes of components. This approach is used in particular in models describing the plastic behavior of soils. However, results are contradictory with a micro–macro approach for Biot's poroelastic model. Additionally, it reduces the number of modes of acoustic waves which rules out its applicability in wave analysis.

(3) *An evolution equation for porosity* typical for a microstructural variable. This sort of model was proposed by Bowen (e.g. [26,27], Bowen's contribution to [10]). It is based on intuitions stemming from chemistry (e.g. evolution equations for the extent of chemical reactions) and some theories of defects (e.g. theory of dislocations with Orowan's equation for the density of dislocations). It does not seem to be appropriate for soils and rocks.

(4) *A balance equation following from the so-called principle of equilibrated pressures* introduced by Goodman and Cowin [28] and modified by Fang [29], Hutter and Kirchner [30] (e.g. for applications in theories of combustion of granulates, abrasion, landslides, avalanches, etc.). This yields a second-order partial differential equation for porosity.

(5) *A balance equation of porosity* (e.g. [11,18]):

$$\frac{\partial \Delta_n}{\partial t} + \operatorname{Div} \mathbf{J} = -\frac{\Delta_n}{\tau}, \quad \Delta_n := n - n_E, \qquad (30)$$

in Lagrangian (i.e. referred to a reference configuration of the skeleton $\mathcal{B}_0$) description. $n$ is the porosity, $n_E$ is the equilibrium porosity which must be given by a constitutive law, $\mathbf{J}$ is the porosity flux which in the simplest case is proportional to the filter velocity, i.e. it vanishes when there is no diffusion (e.g. in thermodynamic equilibria). The right-hand side describes the relaxation ($\tau$ is the relaxation time) which is needed in biomechanical but, most likely, not in geotechnical applications. In the simplest case of large relaxation times ($\tau \to \infty$) this

equation for the linear isotropic case has the form

$$\frac{\partial \Delta_n}{\partial t} + \Phi \operatorname{div}(\mathbf{v}^F - \mathbf{v}^S) = 0, \quad n_E = n_0(1 + \delta e), \tag{31}$$

where $\delta, \Phi$ are constant material parameters. They can be easily related to compressibilities of true materials by means of the same Gedankenexperiments which yield the Gassmann relations for Biot's model (see: [31] as well as further parts of this work). The above equation has the following solution:

$$n = n_0 \left[ 1 + \delta e + \frac{\Phi}{n_0} (e - \varepsilon) \right]. \tag{32}$$

Such changes are always present in Biot's model but not stated explicitly. The reason is that porosity in Biot's model is not an *independent* field. It is given in terms of volumetric strains of both components, $e, \varepsilon$, and the material parameters required by the above equation, $\delta, \Phi$, follow from material parameters present in the model (no additional material parameters for porosity!).

## 2.4. Constitutive relations

In order to construct field equations by means of the above balance equations we need in general constitutive relations for the following *constitutive quantities*:

$$\{ \mathbf{T}^S, \mathbf{T}^F, \hat{\mathbf{p}}, \mathbf{J}, n_E, \hat{n} \}. \tag{33}$$

We have indicated already some simplifications in this set. Instead of the tensor $\mathbf{T}^F$ we need a constitutive relation only for the partial pressure $p^F$, for the porosity balance equation we dispense with the formulation of the constitutive problem and assume relations discussed in the previous section. Consequently, in order to close the system we need constitutive relations of the form

$$\mathbf{T}^S = \mathbf{T}^S(\mathbf{e}^S, \varepsilon, \Delta_n, \mathbf{v}^F - \mathbf{v}^S, \mathbf{a}_r, \operatorname{grad} n),$$

$$p^F = p^F(\mathbf{e}^S, \varepsilon, \Delta_n, \mathbf{v}^F - \mathbf{v}^S, \mathbf{a}_r, \operatorname{grad} n), \tag{34}$$

and for the momentum source which we present in a linear form with respect to vectorial variables

$$\hat{\mathbf{p}} = \hat{\mathbf{p}}_{\text{diff}} - N \operatorname{grad} n - \rho_{12} \mathbf{a}_r, \tag{35}$$

where $\hat{\mathbf{p}}_{\text{diff}}$ is the part determined by the relative velocity $\mathbf{v}^F - \mathbf{v}^S$, $N$ is a material parameter, and $\mathbf{a}_r$ is the relative acceleration (see Section 3.2). It can be shown [21] that only the first term in (35) contributes to the dissipation (8). We discuss the first contribution separately further in this work. In the simplest case $\hat{\mathbf{p}}_{\text{diff}} = \pi(\mathbf{v}^F - \mathbf{v}^S)$, where $\pi$ is a constant permeability coefficient.

The constitutive dependence on $\operatorname{grad} n$ is essential and yields the possibility of the coupling proposed by Biot. The structure of the dependence of stresses on $\Delta_n$ and $\mathbf{v}^F - \mathbf{v}^S$ will be presented in Section 3.2. For the purpose of construction of Biot's model we can omit the dependence on $\Delta_n$. The lack of a dependence on $\operatorname{grad} n$ yields the

following result [19]:

$$\mathbf{T}^S = \mathbf{T}^S(\mathbf{e}^S), \quad p^F = p^F(\varepsilon). \tag{36}$$

This is the *simple mixture* model whose linearization yields Biot's constitutive relations for stresses with $Q = 0$.

In the linear model which follows from (34) and (35) we obtain the following relations for the partial stresses:

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S e \mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon \mathbf{1} - N(n - n_0)\mathbf{1},$$

$$p^F = p_0^F - \rho_0^F \kappa \varepsilon - Qe - N(n - n_0), \tag{37}$$

where $n_0$ is a constant reference value of the porosity, and $\lambda^S, \mu^S, \kappa, Q, N$ are material constants.[3] We have incorporated in the stresses (37) the contribution of $\operatorname{grad} n$ appearing in the momentum source (35). This is possible for a constant coefficient $N$ which is the case in the linear model. We discuss the role of the additional material parameter $N$ further in the section on Gassmann relations. However, it should be stressed that it may vanish in a *linear model*.

Bearing the above relations in mind we see that the set of balance equations (19), (23), (27), (30), for two scalars, two vectors and one tensors form the set of field equations for the fields $\{\mathbf{e}^S, \varepsilon, n, \mathbf{v}^S, \mathbf{v}^F\}$. Clearly, if we introduce the displacement vector $\mathbf{u}^S$ then we can dispense of the tensor equation for $\mathbf{e}^S$, and the momentum balance equation for the skeleton becomes second-order equation for this displacement.

This completes the general remarks on the construction of a two-component model for poroelastic materials. In the following sections we concentrate on Biot's model.

## 3. Remarks on Biot's model

### 3.1. Coupling through partial stresses, simple mixture model

We proceed to discuss question (1) listed in the preliminaries. Already in his first paper in 1941 on consolidation Biot introduced the constitutive coupling between partial stresses in the skeleton and fluid pressure through volume changes of the respective components. We write these relations in the notation commonly used in classical elasticity and fluid mechanics. As indicated in the last section, if we neglect the influence of the gradient of porosity ($N \equiv 0$ in (37)) we obtain for the partial stresses $\mathbf{T}^S$ in the skeleton and $\mathbf{T}^F$ in the fluid

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S e \mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon \mathbf{1}, \quad e = \operatorname{tr} \mathbf{e}^S,$$

$$\mathbf{T}^F = -p^F \mathbf{1}, \quad p^F = p_0^F - \rho_0^F \kappa \varepsilon - Qe, \quad \varepsilon = \frac{\rho_0^F - \rho^F}{\rho_0^F}, \tag{38}$$

---

[3]It is rather unfortunate that the constant $N$ appears in this work in two different meanings. On the one hand, it is identical with the shear modulus $\mu^S$ in some works on Biot's model and, on the other hand, it denotes a material parameter describing the influence of the gradient of porosity. We refer sporadically to the first meaning of this constant in the present work but we use it in the model only in the second meaning.

where $\lambda^S$, $\mu^S$ correspond to the classical Lamé constants, $\kappa$ is the compressibility of the fluid component, and $Q$ is Biot's coupling constant. Of course, these relations are *macroscopic* and, for this reason, material parameters are *effective macroscopic material parameters* which depend on the initial porosity $n_0$.

We quote here another form of these constitutive relations which possesses a particular practical bearing. First of all, we can use the variable $\zeta$, the increment of fluid content, instead of the volume change of fluid, $\varepsilon$. Bearing relation (25) in mind, we obtain

$$\mathbf{T} \approx \mathbf{T}^S + \mathbf{T}^F$$

$$= \mathbf{T}_0 + (\lambda^S + \rho_0^F \kappa + 2Q)e\mathbf{1} + 2\mu^S \mathbf{e}^S - \frac{Q + \rho_0^F \kappa}{n_0}\zeta\mathbf{1},$$

$$\mathbf{T}_0 = \mathbf{T}_0^S - p_0^F \mathbf{1},$$

$$p = \frac{1}{n}p^F$$

$$\approx \frac{1}{n_0}\left(1 - \frac{n - n_0}{n_0}\right)$$

$$\times \left[n_0 p_0 - (Q + \rho_0^F \kappa)e + \frac{\rho_0^F \kappa}{n_0}\zeta\right], \quad p_0 = \frac{p_0^F}{n_0}, \quad (39)$$

where according to (32),

$$\frac{n - n_0}{n_0} = \delta e - \frac{\Phi}{n_0^2}\zeta. \quad (40)$$

If the initial pore pressure, $p_0$, is much smaller than the material parameters $(Q + \rho_0^F \kappa)/n_0$, $\rho_0^F \kappa/n_0^2$ then we can neglect the contribution following from changes of porosity. We obtain finally:

$$\mathbf{T} = \mathbf{T}_0 + Ke\mathbf{1} + 2G\,\mathrm{dev}\,\mathbf{e}^S - C\zeta\mathbf{1}, \quad \mathrm{dev}\,\mathbf{e}^S = \mathbf{e}^S - \frac{1}{3}e\mathbf{1},$$
$$(41)$$

$$K = \lambda^S + \frac{2}{3}\mu^S + \rho_0^F \kappa + 2Q, \quad C = \frac{Q + \rho_0^F \kappa}{n_0}, \quad G = \mu^S,$$

$$p = p_0 - Ce + M\zeta, \quad M = \frac{\rho_0^F \kappa}{n_0^2}. \quad (42)$$

This form shows clearly that stresses consist of equilibrium parts given by the deformation tensor $\mathbf{e}^S$ and noneequilibrium parts given by the increment of fluid contents $\zeta$ (comp. (26)). Consequently, one cannot expect to obtain the field equations based on the above constitutive relations by means of any variational principle. It is long known (e.g. comp. [32]) that a variational principle yields the invariance with respect to time reversal and hence cannot be formulated for irreversible processes. Biot applied the variation of the stress potential to obtain relations (41), (42) but this is, of course, not the variational principle—equations of motion are not following from the stationarity of any functional constructed by means of this potential.

Table 1
Material elastic parameters

| Wilmanski (present work) | $\lambda^S$ | $\mu^S$ | $\kappa$ | $Q$ |
|---|---|---|---|---|
| Stoll [33] | $K$ | $G$ | $M$ | $C$ |
| Biot [7] | $A$ | $N$ | $R$ | $Q$ |
| Biot [34] | $\lambda_c$ | $\mu$ | $M$ | $\alpha$ |
| Bourbie et al. [35] | $\lambda_f$ | $\mu$ | $M$ | $\beta$ |
| Allard [36] | $P$ | $N$ | $R$ | $Q$ |

Biot has been also using in his first work on consolidation [3] a different set of constants but these seem to be abandoned and we shall not quote them here. In Table 1 below we have collected some parameters used in different works on Biot's model.

where $K, G, M, C$ of Table 1 are given by relations (42) and

$$A = \lambda^S, \quad N = \mu^S, \quad R = \rho_0^F \kappa,$$

$$\lambda_c = \lambda^S + \rho_0^F \kappa + 2Q, \quad \mu = \mu^S, \quad \alpha = n_0 \frac{\rho_0^F \kappa + Q}{\rho_0^F \kappa},$$

$$\lambda_f = \lambda^S + \rho_0^F \kappa + 2Q, \quad \beta = n_0 \frac{Q + \rho_0^F \kappa}{\rho_0^F \kappa} \equiv \alpha,$$

$$P = \lambda^S + 2\mu^S \equiv A + 2N. \quad (43)$$

As indicated in Section 1.3, the thermodynamic derivation of constitutive relations for stresses requires a special construction of constitutive relations in the nonlinear case. Namely, we have to include a dependence on higher gradients in order to obtain the coupling term described in the above relations by the constant $Q$.

### 3.2. Dependence on relative accelerations

We turn now our attention to question (2) listed in the preliminaries. As proposed by Biot it is assumed that momentum balance equations contain a contribution of relative accelerations (comp. (1)). This implies that the matrix of partial mass densities is not diagonal. The off-diagonal part is assumed to be symmetric, i.e. this additional contribution is described by a material parameter $\rho_{12}$. It is easy to check that such a contribution violates the principle of material objectivity [13]. However, one may construct a nonlinear model which is materially objective and reduces in the linear limit to the Biot contribution (for details see [21]).

Such a nonlinear objective model with a contribution of relative accelerations is thermodynamically admissible if we add some nonlinear terms to partial stresses and to the free energy. They reflect in the simplest manner the existence of fluctuations of the microstructural kinetic energy caused by the heterogeneity of momentum in the representative elementary volume. The existence of such fluctuations as a result of tortuosity of porous materials

has been indicated by Coussy [37]. There exist some attempts to derive Biot's model with the contribution of relative acceleration by means of Hamilton's principle based on the fluctuation kinetic energy. As the true variational principle does not hold for dissipative systems the dissipation through fluctuation and diffusion is accounted for by a pseudo-potential and a pseudo-variational principle. This does not seem to be the right way of handling irreversible processes, and it is appropriate to rely rather on nonequilibrium thermodynamics. In this section, we present only the main idea in order to specify the position of Biot's model.

In classical mechanics the space of motion is *isometric*, i.e. the distance of two arbitrary points, say $\mathbf{x}_1, \mathbf{x}_2$, must be the same for all reference systems. If the position vectors of these two points in another reference system are described by $\mathbf{x}_1^*, \mathbf{x}_2^*$ then this condition means $|\mathbf{x}_1 - \mathbf{x}_2| = |\mathbf{x}_1^* - \mathbf{x}_2^*|$. This means that there exist a time-dependent orthogonal matrix (*the matrix of rotations*) $\mathbf{O}(t)$ and a time-dependent vector $\mathbf{c}(t)$ (*the vector of translation of the origin*) such that the change of the reference systems is described by the transformation

$$\mathbf{x}^* = \mathbf{O}(t)\mathbf{x} + \mathbf{c}(t), \quad \mathbf{O}^{-1} = \mathbf{O}^{\mathrm{T}} \quad (44)$$

for an arbitrary point $\mathbf{x}$ of the space of motion. This transformation yields rules of the transformation for different quantities appearing in continuum thermodynamics. For instance, scalars do not react on such a transformation and the velocity fields, $\mathbf{v}^{\mathrm{S}}, \mathbf{v}^{\mathrm{F}}$, their time derivatives (accelerations in linear models), the Almansi–Hamel deformation tensor, $\mathbf{e}^{\mathrm{S}}$, and the stress tensor transform in the following way:

$$\mathbf{v}^{\mathrm{S}*} = \mathbf{O}\mathbf{v}^{\mathrm{S}} + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{c}}, \quad \mathbf{v}^{\mathrm{F}*} = \mathbf{O}\mathbf{v}^{\mathrm{F}} + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{c}},$$

$$\dot{\mathbf{O}} = \frac{\mathrm{d}\mathbf{O}}{\mathrm{d}t}, \quad \dot{\mathbf{c}} = \frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t},$$

$$\Longrightarrow \mathbf{v}^{\mathrm{F}*} - \mathbf{v}^{\mathrm{S}*} = \mathbf{O}(\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}), \quad (45)$$

$$\frac{\partial \mathbf{v}^{\mathrm{S}*}}{\partial t} = \mathbf{O}\frac{\partial \mathbf{v}^{\mathrm{S}}}{\partial t} + 2\dot{\mathbf{O}}\mathbf{v}^{\mathrm{S}} + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{c}},$$

$$\frac{\partial \mathbf{v}^{\mathrm{F}*}}{\partial t} = \mathbf{O}\frac{\partial \mathbf{v}^{\mathrm{F}}}{\partial t} + 2\dot{\mathbf{O}}\mathbf{v}^{\mathrm{F}} + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{c}},$$

$$\Longrightarrow \frac{\partial \mathbf{v}^{\mathrm{F}*}}{\partial t} - \frac{\partial \mathbf{v}^{\mathrm{S}*}}{\partial t} = \mathbf{O}\left(\frac{\partial \mathbf{v}^{\mathrm{F}}}{\partial t} - \frac{\partial \mathbf{v}^{\mathrm{S}}}{\partial t}\right) + 2\dot{\mathbf{O}}(\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}), \quad (46)$$

$$\mathbf{e}^{\mathrm{S}*} = \mathbf{O}\mathbf{e}^{\mathrm{S}}\mathbf{O}^{\mathrm{T}}, \quad \mathbf{T}^{\mathrm{S}*} = \mathbf{O}\mathbf{T}^{\mathrm{S}}\mathbf{O}^{\mathrm{T}}. \quad (47)$$

Due to the presence of time derivatives of $\mathbf{O}$ and $\mathbf{c}$ in their transformation rules we say that velocities and accelerations are *not objective* and the relative velocity, the deformation tensor as well as the stress tensor are objective.

The principle of material objectivity requires that constitutive relations are not influenced by transformation (44). This means that, for instance, a classical constitutive

law for elastic materials should transform in the following way:

$$\mathbf{T}^{\mathrm{S}} = \mathscr{T}^{\mathrm{S}}(\mathbf{e}^{\mathrm{S}}) \Longrightarrow \mathbf{T}^{\mathrm{S}*} = \mathscr{T}^{\mathrm{S}}(\mathbf{e}^{\mathrm{S}*})$$

$$\Longrightarrow \mathbf{O}\mathscr{T}^{\mathrm{S}}(\mathbf{e}^{\mathrm{S}})\mathbf{O}^{\mathrm{T}} = \mathscr{T}^{\mathrm{S}}(\mathbf{O}\mathbf{e}^{\mathrm{S}}\mathbf{O}^{\mathrm{T}}). \quad (48)$$

It is important that the constitutive *function* $\mathscr{T}^{\mathrm{S}}$ is the same in untransformed and in transformed relations. It is easy to see that the linear relation for $\hat{\mathbf{p}}$ applied by Biot is not invariant (materially objective) due to the presence of the relative acceleration in the form of the difference (46).

It can be shown [21] that the following *nonlinear* definition of the relative acceleration is objective:

$$\mathbf{a}_r = \left(\frac{\partial}{\partial t} + \mathbf{v}^{\mathrm{S}} \cdot \mathrm{grad}\right)(\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}})$$

$$- [(1 - \mathfrak{z})\,\mathrm{grad}\,\mathbf{v}^{\mathrm{F}} + \mathfrak{z}\,\mathrm{grad}\,\mathbf{v}^{\mathrm{S}}](\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}})$$

$$\Longrightarrow \mathbf{a}_r^* = \mathbf{O}\mathbf{a}_r, \quad (49)$$

where $\mathfrak{z}$ is an arbitrary constitutive parameter, not necessarily constant. Consequently, this acceleration can be used as a constitutive variable similarly to, say, deformation tensor $\mathbf{e}^{\mathrm{S}}$ or the relative velocity $\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}$. Let us remark that the structure of the above definition reminds the well-known structure of objective time derivatives used in rheology.

Thermodynamic analysis presented in [21] yields the following conclusions for the *intrinsic part* of Helmholtz free energy function $\psi$ and for the partial stress tensors $\mathbf{T}^{\mathrm{S}}, \mathbf{T}^{\mathrm{F}}$:

$$\rho\psi = \rho^{\mathrm{S}}\psi^{\mathrm{S}} + \rho^{\mathrm{F}}\psi^{\mathrm{F}} - \frac{\beta}{\Phi}J^{\mathrm{S}}\Delta_n^2 - \frac{1}{2}\rho_{12}(\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}) \cdot (\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}),$$

$$\mathbf{T}^{\mathrm{S}} = \mathbf{T}_B^{\mathrm{S}}(\mathbf{e}^{\mathrm{S}}, \varepsilon) + \beta\Delta_n\mathbf{1} - \mathfrak{z}\rho_{12}(\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}) \otimes (\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}),$$

$$\mathbf{T}^{\mathrm{F}} = -p_B^{\mathrm{F}}(e, \varepsilon)\mathbf{1} - \beta\Delta_n\mathbf{1} - (1 - \mathfrak{z})\rho_{12}(\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}) \otimes (\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}),$$

$$(50)$$

where the contributions $\mathbf{T}_B^{\mathrm{S}}, p_B^{\mathrm{F}}$ lead to the Biot constitutive relations for partial stresses after *linearization*. Hence, neither the Helmholtz free energy $\psi$ nor partial stresses $\mathbf{T}^{\mathrm{S}}$, $\mathbf{T}^{\mathrm{F}}$ depend on the relative acceleration $\mathbf{a}_r$. Solely in the source of momentum (35) this dependence is linear and leads in Biot's model to the contribution with the added mass coefficient $\rho_{12}$.

It is clear that the Helmholtz free energy is *not a potential* for partial stresses even though it is for the internal energy ($= \psi - T(\partial\psi/\partial T)$, T—absolute temperature) and for the entropy ($= -\partial\psi/\partial T$). However, it is interesting to see that the total Helmholtz free energy splits into partial energies, a contribution of nonequilibrium changes of porosity and an interaction term related to the velocity of diffusion. The latter is attributed to the macroscopic influence of fluctuations of microscopic kinetic energy of the fluid.

Let us note that the above form of the Helmholtz free energy is one of a very few examples known in thermodynamics in which it contains nonequilibrium

contributions. This is the reason that it is not a potential for stresses and, simultaneously, it confirms the statement that such models cannot be constructed by any true variational principle.

The above constitutive relations contain not only a nonlinear contribution related to the added mass $\rho_{12}$ but also a linear contribution of the deviation $\Delta_n$ of porosity from its equilibrium value. It can be shown that the material coefficient $\beta$ is so small that this term can safely be neglected in the linear model [16]. However, it plays an important role in the description of the structure of nonlinear waves which we do not discuss in this note.

Values of the coefficient $\rho_{12}$ were investigated in the literature and usually it is assumed that they are related to the porosity. This is rather strange because for the same value of porosity $n_0$ the curvature of channels may vary almost arbitrarily and this means that one cannot expect any relation between $\rho_{12}$ and $n_0$. In spite of this objection, some of these semi-empirical relations seem to correspond rather well with experimental data. We use further the relation proposed by Berryman [38]

$$\rho_{12} = \rho_0^F(1-a), \quad a = \frac{1}{2}\left(\frac{1}{n_0}+1\right) \quad \text{i.e.} \quad a \geqslant 1, \quad \rho_{12} \leqslant 0. \tag{51}$$

We conclude that the contribution of added mass $\rho_{12}$ proposed by Biot is nonobjective but, simultaneously, this nonobjectivity results from neglecting nonlinear terms to the objective relative acceleration. Consequently, it can be accepted in the linear model provided we refer the motion to inertial reference systems. The transformation to non-inertial systems, convenient, for example, in the description of processes in a centrifugal motion, cannot be performed on Biot's equations of motion, and we have to account carefully for the above arguments.

### 3.3. Structure of momentum source, diffusion, tortuosity

Now we discuss question (4) stated in the preliminaries. It was a great discovery of Biot that the momentum source $\hat{\mathbf{p}}$ should be not only dependent on the relative velocity (diffusion) but it should also be variable due to the morphology of pores and the true viscosity of the fluid. This dependence yields a frequency dependence of the bulk permeability (diffusion) coefficient for monochromatic waves. By means of a simple analysis of flows caused by harmonic vibrations in a particular geometry of the microstructure Biot showed that this influence may be neglected in the range of low frequencies where both components move in a synchronized way but it must be accounted for in the range of high frequencies. His formulae for the permeability coefficient contain the true viscosity and some geometric properties of the microstructure. Biot speculated as well that the reason for this dependence is not only viscosity but also a complex pattern of the flow in channels which is attributed to *tortuosity*.

In order to follow Biot's arguments let us rewrite Eqs. (1) in the form appearing in his famous paper on acoustic waves [7] (Eq. (6.6)):[4]

$$\text{div}\,\mathbf{T}^S = \frac{\partial^2}{\partial t^2}(\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}) + \pi\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}),$$
$$\rho_{11} = \rho_0^S - \rho_{12},$$

$$-\text{grad}\,p^F = \frac{\partial^2}{\partial t^2}(\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}) - \pi\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{U}),$$
$$\rho_{22} = \rho_0^F - \rho_{12}. \tag{52}$$

These equations are modified in Biot's next paper [8] in which the coefficient $\pi$ is replaced by the product of this coefficient with the complex function $F(\omega)$. The form of this correction is derived for the microscopic Poiseuille flow in various channels of the porous material and the real and imaginary parts of $F$ are plotted as functions of the frequency. The value of $F$ in the limit of low frequencies is approximately one and the complex form of $F$ must be introduced in the range of high frequencies.

Biot's argument is repeated in many papers on dynamics of porous materials (e.g. see [36], Eqs. (6.52) and (5.64))[5] and there exists an extensive experimental evidence which supports Biot's idea (e.g. see [40] for a brief review).

The problem is that the correction of equations by introducing the frequency-dependent complex permeability, say $\pi F(\omega)$, is mathematically inappropriate. After such a correction, solutions of the equations of motion would have to be complex. In addition, the functions appearing in these equations are defined on different spaces: some of them are time dependent and some other frequency dependent and this is, of course, mathematically not correct. Papers concerning this subject avoid the problem by writing the Fourier transform of Eqs. (52) in which the variable of the Fourier transformation is identified with the frequency appearing in $F$.

The above remarks indicate the necessity of a correction in the formulation of the problem and this, as a matter of fact, was already suggested by Biot himself. Namely, the

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[4]Biot writes only the $x$-component of this equation but his remark in Section 3 makes clear that the other two components should be analogous. The components of the stress tensor $\mathbf{T}^S$ are denoted: $(\sigma_x, \tau_y, \tau_z)$ on the $x$-plane and the notation for other components is not introduced. Instead of our coefficient $\pi$ Biot uses $b$. The partial pressure $p^F$ is denoted by $-s$.

[5]The paper of Bedford et al. [39] contains a typical argument concerning the frequency dependence. As they write after relation (2) coefficients $\pi$ and $\rho_{12}$ ($b$ and $-c$ in their notation) are *constitutive constants*. Then, in Section 1, the following simple motion is considered by the authors. The one-dimensional displacements $u_s$ and $u_f$ are assumed to be independent of the spacial variable and to have the form $u_s = e^{i\omega t}$ and $u_f = Ue^{i\omega t}$ where $U$ is a complex constant. The substitution in Biot's equations yields in our notation the compatibility conditions (relations (6) and (7) in the above quoted paper)

$$\pi = \rho_0^F\omega\,\text{Im}\,\frac{U}{U-1}, \quad \rho_{12} = \rho_0^F\,\text{Re}\,\frac{U}{U-1}.$$

Hence, the "constitutive constants" are no longer constant and, if we inverse the Fourier transform ($\omega \to t$), constants become functions of time!

influence of relative velocities in the macroscopic model should have a form reflecting viscous-like effects following from obstacles for the motion of the real fluid in channels. It means that, similarly to viscoelastic materials, one should expect a kind of hereditary integral describing the momentum source.

In spite of their practical importance explicit relations for the bulk permeability $\pi$ are not needed in the macroscopic construction of such a model. They may be adjusted to experiments or selected by means of some analysis of microstructure in particular problems (e.g. see: [41]). It is only essential in the linear model to account for their changes in time as a dynamic process goes on. This can be done in the same way as it is done for viscoelastic properties of solids (e.g. [42]). We proceed to construct such a model of momentum sources.

To this end, the second law of thermodynamics indicates the following dissipation inequality:

$$\hat{\mathbf{p}}_{\text{diff}} \cdot \mathbf{w} \geqslant 0, \quad \mathbf{w} := \mathbf{v}^{\text{F}} - \mathbf{v}^{\text{S}}. \tag{53}$$

Consequently, $\hat{\mathbf{p}}_{\text{diff}}$ must be odd in the relative velocity. Within a linear model of isotropic materials it has then the following form for a *constant* relative velocity $\mathbf{w}$,

$$\hat{\mathbf{p}}_{\text{diff}}(\mathbf{w}, t) = \pi(t)\mathbf{w}, \tag{54}$$

which indicates a smallness of the relative velocity. Knowledge of this single-step response function $\pi(t)$ is sufficient to predict an output $\hat{\mathbf{p}}_{\text{diff}}$ for an arbitrary constant input $\mathbf{w}$. Let us note that the time dependence of the permeability coefficient $\pi$ should be invariant with respect to the translation of the origin of the time axis. This means that $t$ is the time lag since application of the constant relative velocity $\mathbf{w}$. An input $\mathbf{w}(t) = \mathbf{w}_0 H(t - t_0)$, where $H(t - t_0)$ is the Heaviside function would lead to the output $\hat{\mathbf{p}}_{\text{diff}} = \pi(t - t_0)\mathbf{w}_0$. Consequently, for the two-step history

$$\mathbf{w}(t) = H(t - t_1)\Delta\mathbf{w}_1 + H(t - t_2)\Delta\mathbf{w}_2, \tag{55}$$

the output must have the following form:

$$\hat{\mathbf{p}}_{\text{diff}}(t) = \pi_1(t, t_1, t_2)\Delta\mathbf{w}_1 + \pi_2(t, t_1, t_2)\Delta\mathbf{w}_2. \tag{56}$$

This expression is valid for all constant increments of the relative velocity. Hence, it holds in particular for $\Delta\mathbf{w}_2 = 0$. Then the momentum source must be $\pi(t - t_1)\Delta\mathbf{w}_1$. In the same way we identify the second contribution and it follows

$$\mathbf{w}(t) = \sum_{\alpha=1}^{2} H(t - t_\alpha)\Delta\mathbf{w}_\alpha \Longrightarrow \hat{\mathbf{p}}_{\text{diff}}(t) = \sum_{\alpha=1}^{2} \pi_\alpha(t - t_\alpha)\Delta\mathbf{w}_\alpha. \tag{57}$$

Certainly, the same argument applies for an arbitrary number of steps. In rheology this is called Boltzmann's superposition principle. By means of continuity assumptions which are similar to those of the theory of viscoelasticity we can approximate by step history the response to an arbitrary integrable history of the relative

velocity. We have then

$$\mathbf{w}(t) = \int_{-\infty}^{t} H(t - s)\,d\mathbf{w}(s) \Longrightarrow \hat{\mathbf{p}}_{\text{diff}}(t) = \int_{-\infty}^{t} \pi(t - s)\,d\mathbf{w}(s), \tag{58}$$

which are the so-called Stieltjes convolutions. As we assume invariance with respect to translation the initial point of the relative motion can be chosen arbitrarily. Therefore, it is convenient to set the lower limit of integration at $-\infty$.

Integration by parts and change of variables yields

$$\hat{\mathbf{p}}_{\text{diff}}(t) = \pi(0)\mathbf{w}(t) + \int_{0}^{\infty} \dot{\pi}(s)\mathbf{w}(t - s)\,ds, \quad \dot{\pi}(s) := \frac{\partial\pi}{\partial s}. \tag{59}$$

This constitutive relation contains the memory functional typical for a viscous effect. The instantaneous reaction is characterized by the initial permeability $\pi(0)$. After Biot's work [7] it is argued in the literature of the subject that this is the only contribution in cases of monochromatic waves of low frequency.

We demonstrate here only some simple properties of the above relation but it is obvious that the relation for the momentum source can be easily extended on nonlinear models. Such extensions are known for memory materials and we shall not present any details in this note.

Let us write relation (59) for the harmonic disturbance of the frequency $\omega$. We obtain immediately

$$\mathbf{w} = \mathbf{w}^*(\mathbf{x})e^{-i\omega t}, \quad \hat{\mathbf{p}}_{\text{diff}} = \hat{\mathbf{p}}_{\text{diff}}^*(\mathbf{x})e^{-i\omega t}$$

$$\Longrightarrow \hat{\mathbf{p}}_{\text{diff}}^* = \left(-i\omega \frac{1}{\pi(0)} \int_{0}^{\infty} \pi(s)e^{i\omega s}\,ds\right)\pi(0)\mathbf{w}^*. \tag{60}$$

The transform of the permeability coefficient in parentheses can now be identified with Johnson's relation [40,41] or any other relation proposed for the permeability as a function of frequency. This contribution appears in the Fourier transformed momentum balance equations rather than in the original momentum balance equations which hold for an arbitrary time-dependent input.

The above hereditary form of the momentum source is, as already indicated, attributed to the viscosity of the true fluid and to the tortuosity. However, the influence of tortuosity is assumed to lead also to an added mass effect and, consequently, to the nondiagonal form of the matrix of mass densities. This yields contributions of the relative accelerations which were discussed in the previous section. We return to this interpretation of the added mass in the section on monochromatic waves.

### 3.4. Microstructural (Gassmann-like) relations

In this section we answer question (3) formulated in the preliminaries and, simultaneously, we relate some material parameters to material constants of true materials. In particular, we demonstrate that the model with dependence on the gradient of the porosity is meaningful, i.e. the material constant $N$ can indeed be estimated. However,

considerations of this section do not form an immanent part of the macroscopic model. The latter is fully described in previous sections and its material parameters, say, $\lambda^S, \mu^S, \rho_0^F \kappa, Q, N, \pi(t), \delta, \Phi$ or, equivalently, $K, G, M, C, N, \pi(t), \delta, \Phi$ do not have to be found from any theoretical considerations. Such theoretical considerations are possible only for some of these parameters and only for some materials.

There are some attempts to derive macroscopic equations of poroelasticity from microstructure by use of multiscaling and, consequently, by the method of homogenization. One of the first attempts of this sort is the work of Burridge and Keller [43]. Their results yield the structure of the macroscopic model which coincides with Biot's model but relations for material parameters are not effective. Some further attempts (e.g. [44]) are very sophisticated from the mathematical point of view but they do not seem to produce results of importance in engineering applications, either.

For granular materials there exists a procedure of estimating parameters related to purely volumetric deformations. This procedure stemming from the averaging method over a REV yields relations between $\lambda^S + \frac{2}{3}\mu^S$ or $K$, $\rho_0^F \kappa$, or $M$, $Q$ or $C, N, \delta, \Phi$ and true compressibilities of substances, $K_s, K_f$ of the solid frame and fluid components, respectively, as well as the initial porosity, $n_0$. Certainly, any other combination of macroscopic parameters can then also easily be calculated.

Such relations were considered first by Gassmann [45] and the Gedankenexperiments from which one can systematically derive Gassmann's relations for porous materials were proposed by Biot and Willis [46]. We follow here the considerations presented in [31]. Let us mention that a similar method of derivation used in these considerations has recently independently been proposed by Cheng and Abousleiman [47] (comp. also [48]).

Description of granular materials by means of a macroscopic model is particularly easy when a deformation is homogeneous, spherically symmetric and the mechanical reactions of the system reduce to pressures. Such a system is considered in this section.

In our notation we have for such a deformation

$$\mathbf{e}^S = \tfrac{1}{3}e\mathbf{1}, \quad p^S = -\tfrac{1}{3}\mathrm{tr}\,\mathbf{T}^S, \quad p^F = -\tfrac{1}{3}\mathrm{tr}\,\mathbf{T}^F \tag{61}$$

and the *macroscopic constitutive relations* for partial pressures are as follows (comp. (37)):

$$p^S - p_0^S = -(\lambda^S + \tfrac{2}{3}\mu^S)e - Q\varepsilon + N(n - n_0),$$

$$p^F - p_0^F = -\rho_0^F \kappa\varepsilon - Qe - N(n - n_0). \tag{62}$$

Field equations, say (52), are in homogeneous equilibrium identically satisfied, and the balance equation of porosity reduces to relation (32). We shall later return to the latter.

As a boundary condition in the static case we have the *equilibrium condition of* the full pressure change with a given excess (external) pressure $\Delta p$, i.e.

$$\Delta p = (p^S - p_0^S) + (p^F - p_0^F). \tag{63}$$

Clearly the problem to find volume changes $e, \varepsilon$, partial pressures $p^S$, $p^F$, and porosity $n$ following the loading $\Delta p$ cannot be solved yet even if we account for the relation for porosity (32). We have only four equations at the disposal and we need an additional equation specifying, for instance, a flow of the fluid through the boundary of the medium (the second boundary condition) provided the problem remains homogeneous. This is natural for a two-component system and we present such equations in the sequel.

However, we proceed with the closure of the above problem in a different way [31]. Namely we specify properties of the microstructure and relations between microscopic and macroscopic models. Then the problem can be solved and additional equations mentioned above serve solely the purpose of relating microscopic and macroscopic properties (compatibility conditions!). Such an approach is possible for static problems due to the homogeneity. Then, as mentioned above, solutions of partial momentum equations are trivial and the problem becomes algebraic.

The microscopic model for spherical deformations is specified as follows. Volume changes on the microscopic level are denoted by $e^R$, $\varepsilon^R$. Then $e^R$ describes relative true volume changes of grains (the solid frame), and $\varepsilon^R$ describes changes of the true mass density of the fluid in channels of the granular material. The corresponding pressures are denoted by $p^{SR}$, $p^{FR}$ and, of course, the latter is identical with the pore pressure $p_f$. For these quantities the following *microscopic constitutive relations* are assumed:

$$p^{SR} - p_0^{SR} = -K_s e^R, \quad p^{FR} - p_0^{FR} = -K_f \varepsilon^R. \tag{64}$$

In these relations $K_s$, $K_f$ denote real (true) compressibility moduli of the solid (granule) and fluid components, respectively. In contrast to material parameters of the macroscopic model, these can be measured independently of a current morphology of the granular material. They are not influenced by porosity, cohesivity, or any other property characteristic for the texture of the medium.

The above presented macroscopic and microscopic properties are related through two sets of compatibility conditions. On the one hand, we have *dynamic compatibility relations* which for pressures have the form

$$p^S = (1 - n)p^{SR}, \quad p_0^S = (1 - n_0)p_0^{SR},$$

$$p^F = np^{FR}, \quad p_0^F = n_0 p_0^{FR}. \tag{65}$$

On the other hand, relations between partial and true mass densities (4)

$$\rho^S = (1 - n)\rho^{SR}, \quad \rho_0^S = (1 - n_0)\rho_0^{SR},$$

$$\rho^F = n\rho^{FR}, \quad \rho_0^F = n_0 \rho_0^{FR} \tag{66}$$

and solutions of partial mass balance equations (comp. (18) and (23) for the macroscopic model)

$$\rho^S = \rho_0^S (1+e)^{-1}, \quad \rho^F = \rho_0^F (1+\varepsilon)^{-1},$$

$$\rho^{SR} = \rho_0^{SR} (1+e^R)^{-1}, \quad \rho^{FR} = \rho_0^{FR} (1+\varepsilon^R)^{-1}, \qquad (67)$$

yield immediately the following *geometrical compatibility relations*:

$$e = e^R + \frac{n - n_0}{1 - n_0}, \quad \varepsilon = \varepsilon^R - \frac{n - n_0}{n_0}. \qquad (68)$$

Consequently, for the nine unknown quantities of spherical homogeneous deformations

$$\{e, \varepsilon, p^S, p^F, e^R, \varepsilon^R, p^{SR}, p^{FR}, n\}, \qquad (69)$$

we have nine equations at the disposal: 1 equilibrium condition (63), two macroscopic constitutive relations (62), two microscopic constitutive relations (64), two dynamic compatibility relations (65), and two geometrical compatibility conditions (68). This simple algebraic problem can be solved.

If we solve the problem for changes of porosity [31] and compare the result with the relation for porosity (32) we obtain the following relations for the material parameters $\delta, \Phi$:

$$\delta = \frac{K_V - K}{n_0 (K_s - K_f)}, \quad \Phi = \frac{(Q + \rho_0^F \kappa) - n_0 K_f}{K_s - K_f}, \qquad (70)$$

$$K_V := (1 - n_0) K_s + n_0 K_f. \qquad (71)$$

$K_V$ is the so-called Voigt compressibility modulus. Hence, once the material parameters $K, \rho_0^F \kappa$ and $Q$ are determined in terms of $K_s, K_f$ and $n_0$ the relation for porosity is fully explicit and does not require any additional material parameters.

> This result shows that the porosity is not constant in Biot's model but its changes are described by other fields—namely by $e$ and $\varepsilon$ (or $\zeta$), and parameters of the model.

According to the work of Biot and Willis [46] one can consider three boundary value problems for the above homogeneous case. They are called *jacketed undrained*, *jacketed drained* and *unjacketed Gedankenexperiments*.

The conditions defining these Gedankenexperiments are as follows:

(1) The jacketed undrained experiment is performed on a sample of which the boundary is impermeable, i.e.

$$\zeta = 0 \Rightarrow e = \varepsilon; \qquad (72)$$

it means that there is no flow through the boundary and, consequently, macroscopic volume changes of both components must be equal.

(2) The jacketed drained experiment of which the boundary is permeable, i.e. there is a drainage of the sample connecting the fluid component directly with the external world. Then the excess loading $\Delta p$ is carried only by the skeleton

$$p^{FR} = p_0^{FR} \Rightarrow \varepsilon^R = 0. \qquad (73)$$

(3) The unjacketed experiment in which the sample is immersed in the fluid and the pore pressure and the external fluid pressure must be the same, i.e.

$$p^{FR} - p_0^{FR} = \Delta p, \qquad (74)$$

where $\Delta p$ is a given excess pressure.

These tests are called Gedankenexperiments because they do not have to be really performed but they are principally possible under ideal conditions (e.g. when capillary effects, viscosity, temperature changes, etc. can be neglected), and, simultaneously, they lead to relations between physical properties—in our case, between macroscopic and microscopic material parameters.

It is commonly assumed that in addition to microscopic tests which deliver compressibilities $K_s, K_f$ one can measure the so-called *drained compressibility modulus $K_d$*. This is defined as the negative fraction of the excess pressure $\Delta p$ to the macroscopic volume change $e$ in the drained jacketed test. Consequently, we obtain four additional relations which together with the nine equations listed above form a set of 13 equations for nine unknowns (69). Hence, the system is overdetermined but can be solved provided four parameters of the problem are related to other parameters.

The *jacketed undrained* test described by (72) yields immediately the *first compatibility relation* between material parameters

$$\mathscr{C}_1 := C + \frac{K_f (K - K_s) - N(K - K_V)}{n_0 (K_s - K_f)} = 0. \qquad (75)$$

The definition of the jacketed drained experiment yields the *second compatibility relation*

$$\mathscr{C}_2 := n_0 - \frac{C}{M} - \frac{K_b}{K_s} \frac{1 - (1 - n_0)(K_s/K_b)}{1 - [(1 - n_0)/n_0][NC/K_b M]}$$
$$\times \left\{ 1 - \frac{N(K - n_0 C)}{n_0 M K_b} \right\} = 0,$$

$$K_b := K - \frac{C^2}{M}. \qquad (76)$$

Simultaneously, the definition of the drained compressibility modulus leads to the following *third compatibility relation*:

$$\mathscr{C}_3 := K - K_d - C \frac{C - N}{M - (N/n_0)} = 0. \qquad (77)$$

Finally, for the *unjacketed* test, we obtain the *fourth compatibility relation*:

$$\mathscr{C}_4 := \left( 1 - \frac{K}{K_W} \right) \left( M - C - N \frac{1 - n_0}{n_0} \frac{C}{K_s} \right)$$
$$+ \left( 1 - \frac{C}{K_W} \right) \left( K - C - N \frac{1 - n_0}{n_0} \left( 1 - \frac{K}{K_s} \right) \right) = 0,$$

$$\frac{1}{K_W} := \frac{1 - n_0}{K_s} + \frac{n_0}{K_f}. \tag{78}$$

The set of relations

$$\{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\} = 0 \tag{79}$$

forms four equations for four material parameters $\{K, C, M, N\}$, or equivalently for $\{\lambda^S + \frac{2}{3}\mu^S, \rho_0^F \kappa, Q, N\}$ in terms of the material parameters $\{K_s, K_f, K_d\}$ and the initial porosity $n_0$. It is nonlinear and, consequently, possesses more than one solution. As it cannot be solved analytically we present below some typical numerical results. For the data used in the example there exist two physically reasonable solutions in which all four quantities $K, M, C, N$ are real and nonnegative.

However, one analytical solution can be constructed in the fully general case. Namely, substitution of the parameter $N = 0$ in $\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_4\} = 0$ yields the following set of equations:

$$K - K_s + n_0 C \frac{K_s - K_f}{K_f} = 0, \quad K_d - K + \frac{C^2}{M} = 0,$$

$$K\left(1 - \frac{C}{K}\right) - C + M - \frac{M K_d}{K_W} = 0. \tag{80}$$

This set possesses a real positive solution which can be found analytically and it has the form

$$K = \frac{(K_s - K_d)^2}{(K_s^2/K_W) - K_d} + K_d, \quad C = \frac{K_s(K_s - K_d)}{(K_s^2/K_W) - K_d},$$

$$M = \frac{K_s^2}{(K_s^2/K_W) - K_d}. \tag{81}$$

These are the famous *Gassmann relations* appearing in Biot's model (e.g. [33]). It is rather amazing that they satisfy *identically* the equation $\mathscr{C}_2 = 0$ with $N = 0$. Conse-

quently $\{K, C, M\}$ given by (81) and $N = 0$ constitute a solution of the full system of compatibility conditions. It yields the important conclusion that

> Biot's model of which the material parameters satisfy the Gassmann relations is thermodynamically admissible in spite of the fact that its constitutive relations for partial stresses do no contain a dependence on the porosity gradient.

This property follows solely for the linear model in which the contribution of the porosity gradient can be incorporated in the constitutive relations for the stresses and, simultaneously, for which the conditions of the Gedankenexperiments can be satisfied.

As mentioned above the set of equations $\{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\} = 0$ for material parameters $\{K, M, C, N\}$ cannot be solved analytically. However, it is a rather simple algebraic system which can be handled numerically with any standard package. Below we show such numerical results calculated by means of the package MAPLE 7.0. We have chosen the following data for these calculations

$$K_s = 48 \times 10^9 \, \text{Pa}, \quad K_f = 2.25 \times 10^9 \, \text{Pa}, \quad K_d = \frac{K_s}{1 + g n_0}, \tag{82}$$

where the last empirical relation is proposed after Geertsma (e.g. see: [49] where it is claimed that $g = 50$ gives a good fit with experimental data for many soils).

It should be mentioned that numerical values of the constant $N$ lie below these of the coupling parameter $Q$. The latter, as seen in Fig. 1, is much smaller than both compressibility coefficients, $K, M$. Consequently, the approximation with both constants $Q$ and $N$ equal to zero may lead in some cases to reasonable results. We return to this problem discussing acoustic properties of *simple*
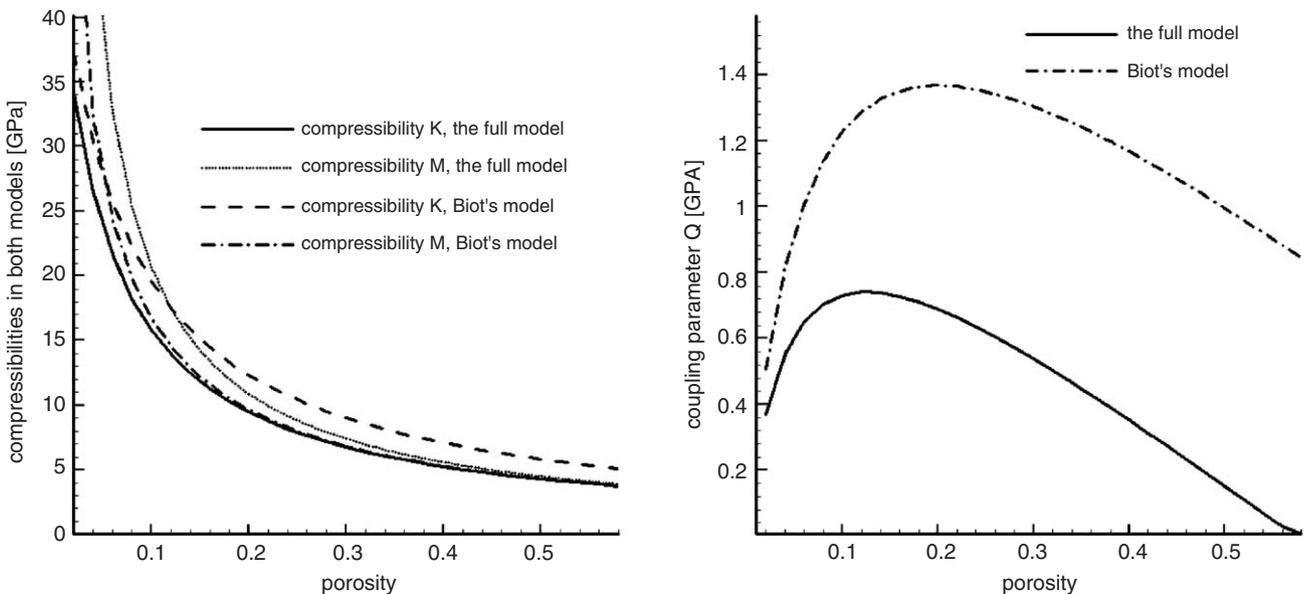


Fig. 1. Comparison of the compressibilities $K, M$ (left) and of the coupling parameter $Q$ (right) for the full model and Biot's model with Geertsma parameter $g = 50$ [31].
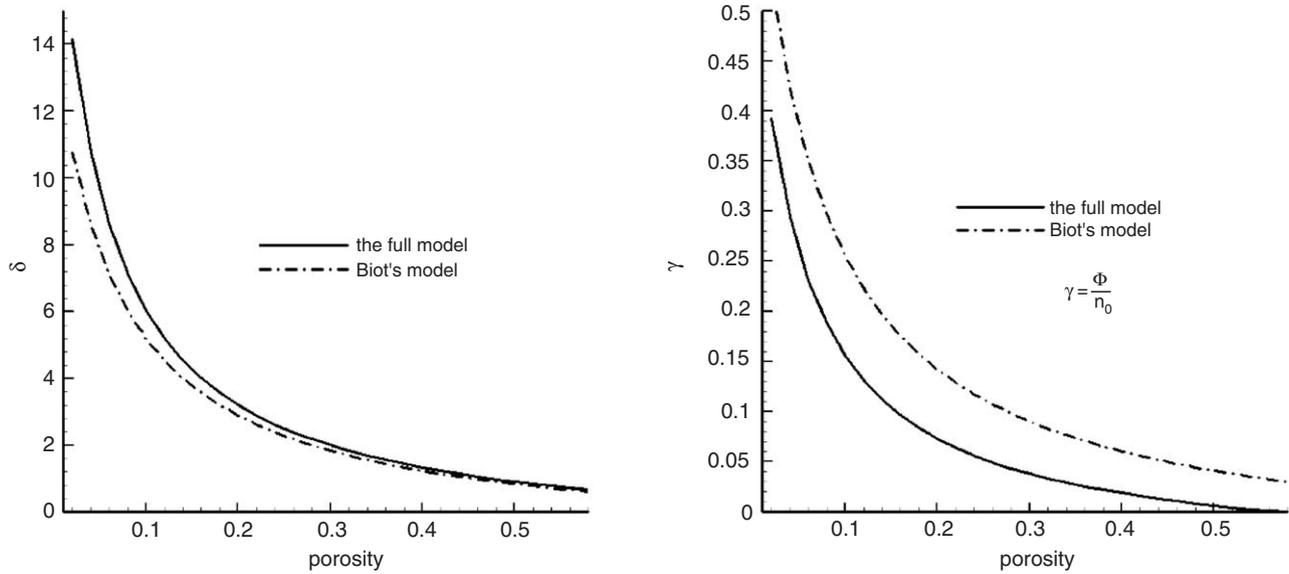
Fig. 2. Coefficients of porosity relation: equilibrium part $\delta$, and nonequilibrium part $\gamma = \Phi/n_0$ with Geertsma parameter $g = 50$ [31].

*mixtures* for which these two constants are assumed to be zero. Incidentally, such a model would not satisfy relations describing Gedankenexperiments and, for this reason, can be only treated as an approximation.

On the left panel of Fig. 1 we compare directly compressibilities $K, M$ for both models. Obviously the compressibility modulus of the skeleton, $K$, is higher (app. 20%) for Biot's model than it is for the full model but it is the other way around for the compressibility of the fluid, $M$.

The coupling coefficient, $Q$, agrees qualitatively for both models and for the porosity $n_0 < 0.58$ (Fig. 1, right panel) but the quantitative differences are essential. This coupling is much stronger in Biot's model and it becomes negative in the full model for $n_0 \gtrsim 0.58$. This means that an increment of pressure would lead to an increment of volume (i.e. a nonconvex potential of stresses) which, according to classical arguments of continuum mechanics, yields an instability.

Further results for different material constants and another choice of the Geertsma parameter can be found in [31]. It is interesting to check the numerical results for the coefficients $\delta$ and $\Phi$ describing the behavior of the current porosity $n$ (see relation (32)). These are shown in Fig. 2. It is clear that equilibrium changes of porosity proportional to volume changes of the skeleton $e$ are much bigger than nonequilibrium changes proportional to the difference of volume changes $e - \varepsilon$ (i.e. to the increment of fluid content $\zeta$). The parameter $\delta$ is app. 20 times bigger than the nonequilibrium parameter $\Phi/n_0$. Changes of porosity are apparently more significant in the range of small initial porosities than for large initial porosities.

It is rather unfortunate that the shear modulus $\mu^S$ cannot be determined by a similar simple Gedankenexperiment. There exist some attempts to relate this macroscopic parameter to properties of the microstructure by some shearing experiment but results are strongly dependent on the choice of the microstructural configuration, shape and size of granule and many other parameters (e.g. see [49]).

## 4. Acoustic waves

### 4.1. Propagation of wave fronts

Acoustic wave analysis based on Biot's model of porous materials is the main subject of investigation in this field of research. Biot's model is not a particularly good tool to describe the behavior of soils which, in most cases—such as behavior of clays, landslides and avalanches, damage of rocks, etc., yield irreversible deformations. It is applicable in some problems of biomechanics but, in this field, one has to account usually for large deformations—for instance, the deformation of lungs is of the order of 70%, the deformation of callus in the bone healing process reaches 20%. Such deformations are not described by Biot's model. However, it is a very efficient and reliable model for the description of acoustic waves of small amplitude which appear in nondestructive testing of materials and in sonar methods. For this reason there is a vast literature on this subject and there are many books dealing with acoustic waves in poroelastic media (e.g. [46–48, 34]). In this note, we aim to present only a few chosen problems of acoustics which do not seem to achieve much attention in the works on this subject. In particular, this concerns the influence of parameters characteristic for Biot's model: the coupling of stresses (parameter $Q$) and the added mass effect (parameter $\rho_{12}$). We omit the problems of memory effects related to the hereditary character of momentum sources as well as the relaxation of porosity. The former is extensively

discussed in the literature, the latter has not been investigated at all.

Acoustic waves in continua are related to the propagation of a nonmaterial singular surface—the *wave front* $\mathscr{S}$—on which acceleration fields are discontinuous but velocity fields remain continuous. If the latter does not hold we deal either with shock waves or with vortex sheets.

Let us assume that the instantaneous geometry of the front $\mathscr{S}$ is given by the following equation:

$$f(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathscr{B}_t = \mathbf{f}^S(\mathscr{B}_0, t) \subset \mathfrak{R}^3, \tag{83}$$

which is at least of class $\mathscr{C}^2$ with respect to $\mathbf{x}$, and of the class $\mathscr{C}^1$ with respect to time $t$. $\mathscr{B}_t$ denotes the current configuration of the skeleton. Smoothness requirements mean that changes of the normal vector to the surface are differentiable, and changes of curvatures are continuous. Simultaneously, there exists a smooth speed of propagation of the surface. In order to see these properties we use the identity

$$\mathrm{d}f \equiv \mathrm{d}\mathbf{x} \cdot \mathrm{grad}\, f + \mathrm{d}t\, \frac{\partial f}{\partial t} = 0. \tag{84}$$

As the gradient of $f$ is orthogonal to the surface ($f$ is constant along the surface, i.e. the vector $\mathrm{grad}\, f$ may possess solely an orthogonal component) we can define a unit normal vector by the relation

$$\mathbf{n} := \frac{\mathrm{grad}\, f}{|\mathrm{grad}\, f|}. \tag{85}$$

The second gradient of $f$ is related to curvatures. Bearing relation (84) in mind we obtain for the speed of propagation

$$c := \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \mathbf{n} = -\frac{\partial f / \partial t}{|\mathrm{grad}\, f|}. \tag{86}$$

It is easy to see that relation (83) does not impose any conditions on the tangential component of the velocity of the surface. This means that the kinematics of slip motions cannot be described by such a relation. However, this is immaterial in the theory of waves. If the speed of propagation $c$ is given, then relation (86) is the nonlinear differential equation for the function $f$:

$$\frac{\partial f}{\partial t} + c|\mathrm{grad}\, f| = 0. \tag{87}$$

With an appropriate initial condition for the position of the front (i.e. $f(\mathbf{x}, t = 0)$-given) this equation forms a nonlinear Cauchy problem.

We proceed to present an example of analysis of weak discontinuity (acoustic) waves for Biot's model as well as the "simple mixture" model in which both the coupling $Q$ and the tortuosity coefficient $(a - 1)$ are assumed to be zero. Similarly to Biot's model the latter model has already a rather extensive literature (for reviews of results, see: [15,50,51]). We rewrite Biot's equations in slightly modified form. For the fields $\mathbf{v}^S$, $\mathbf{v}^F$, $\mathbf{e}^S$, $\varepsilon$, we

have the field equations

$$\rho_{11}\frac{\partial \mathbf{v}^S}{\partial t} + \rho_{12}\frac{\partial \mathbf{v}^F}{\partial t} = \lambda^S \,\mathrm{grad}\,\mathrm{tr}\,\mathbf{e}^S + 2\mu^S \,\mathrm{div}\,\mathbf{e}^S + Q\,\mathrm{grad}\,\varepsilon + \hat{\mathbf{p}},$$

$$\rho_{22}\frac{\partial \mathbf{v}^F}{\partial t} + \rho_{12}\frac{\partial \mathbf{v}^S}{\partial t} = \kappa\rho_0^F \,\mathrm{grad}\,\varepsilon + Q\,\mathrm{grad}\,\mathrm{tr}\,\mathbf{e}^S - \hat{\mathbf{p}}, \tag{88}$$

where

$$\frac{\partial \mathbf{e}^S}{\partial t} = \mathrm{sym}\,\mathrm{grad}\,\mathbf{v}^S, \quad \frac{\partial \varepsilon}{\partial t} = \mathrm{div}\,\mathbf{v}^F,$$

$$\rho_{11} = \rho_0^S[1 - r(1 - a)], \quad \rho_{12} = r(1 - a)\rho_0^S, \quad \rho_{22} = ra\rho_0^S,$$

$$a = \frac{1}{2}\left(\frac{1}{n_0} + 1\right), \quad r = \frac{\rho_0^F}{\rho_0^S} \tag{89}$$

and the source $\hat{\mathbf{p}}$ depends only on the relative velocity.

We begin the analysis of this system by proving its hyperbolicity. To this aim we consider the propagation of the front, $\mathscr{S}$, of a weak discontinuity wave, i.e. of a singular surface on which

$$[[\mathbf{v}^S]] = 0, \quad [[\mathbf{v}^F]] = 0, \tag{90}$$

where $[[\ldots]]$ denotes the jump of the quantity across $\mathscr{S}$. On such a surface accelerations may be discontinuous and we call their jumps the *amplitudes of discontinuity*

$$\mathbf{a}^S := \left[\left[\frac{\partial \mathbf{v}^S}{\partial t}\right]\right], \quad \mathbf{a}^F := \left[\left[\frac{\partial \mathbf{v}^F}{\partial t}\right]\right]. \tag{91}$$

Then the following compatibility conditions hold:[6]

$$[[\mathrm{grad}\,\mathbf{v}^S]] = -\frac{1}{c}\,\mathbf{a}^S \otimes \mathbf{n}, \quad [[\mathrm{grad}\,\mathbf{v}^F]] = -\frac{1}{c}\,\mathbf{a}^F \otimes \mathbf{n},$$

$$[[\mathrm{grad}\,\mathbf{e}^S]] = -\frac{1}{c}\left[\left[\frac{\partial \mathbf{e}^S}{\partial t}\right]\right] \otimes \mathbf{n}, \quad [[\mathrm{grad}\,\varepsilon]] = -\frac{1}{c}\left[\left[\frac{\partial \varepsilon}{\partial t}\right]\right]\mathbf{n}, \tag{92}$$

where $c$ is the speed of propagation of the surface $\mathscr{S}$ and $\mathbf{n}$ its unit normal vector. The latter gives the direction of the propagation of the wave. The sign $\otimes$ denotes the tensor product. Bearing (89) in mind we obtain immediately

$$[[\mathrm{grad}\,\mathbf{e}^S]] = \frac{1}{2c^2}(\mathbf{a}^S \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{a}^S) \otimes \mathbf{n},$$

$$[[\mathrm{grad}\,\varepsilon]] = \frac{1}{c^2}(\mathbf{a}^S \cdot \mathbf{n})\mathbf{n}. \tag{93}$$

We evaluate the jump of the field equations (88) on the surface $\mathscr{S}$. This immediately yields

$$(\rho_{11}c^2\mathbf{1} - \lambda^S\mathbf{n} \otimes \mathbf{n} - \mu^S(\mathbf{1} + \mathbf{n} \otimes \mathbf{n}))\mathbf{a}^S$$
$$+ (\rho_{12}c^2\mathbf{1} - Q\mathbf{n} \otimes \mathbf{n})\mathbf{a}^F = 0,$$

---

[6]The behavior of various kinematic quantities on singular surfaces has been studied since 150 years and the modern theory follows the way proposed by Hadamard. We use a particular case of these kinematic compatibility conditions following with the assumption of continuity of both motions and velocities (e.g. [15]).

$$(\rho_{12}c^2\mathbf{1} - Q\mathbf{n}\otimes\mathbf{n})\mathbf{a}^{\mathrm{S}} + (\rho_{22}c^2\mathbf{1} - \kappa\rho_0^{\mathrm{F}}\mathbf{n}\otimes\mathbf{n})\mathbf{a}^{\mathrm{F}} = 0. \qquad (94)$$

This is clearly an eigenvalue problem. We say that system (88) is *hyperbolic* if the eigenvalues $c$ are real and the corresponding eigenvectors $[\mathbf{a}^{\mathrm{S}}, \mathbf{a}^{\mathrm{F}}]$ are linearly independent. We prove that this is indeed the case.

It is important to observe that the above eigenvalue problem is *independent* of the momentum source $\hat{\mathbf{p}}$. As this source is the only quantity which contributes to the dissipation it means that the propagation of the wave fronts is independent of the diffusion, i.e. the speeds of propagation are the same for thermodynamic equilibrium and nonequilibrium processes. We see further that this property does not extend to attenuations.

It is convenient to separate the transversal and longitudinal parts of problem (94). The *transversal* part follows if we take the scalar product of Eqs. (94) with a vector $\mathbf{n}_\perp$ perpendicular to $\mathbf{n}$. We obtain

$$(\rho_{11}c^2 - \mu^{\mathrm{S}})a_\perp^{\mathrm{S}} + \rho_{12}c^2 a_\perp^{\mathrm{F}} = 0,$$

$$\rho_{12}a_\perp^{\mathrm{S}} + \rho_{22}a_\perp^{\mathrm{F}} = 0,$$

$$a_\perp^{\mathrm{S}} := \mathbf{a}^{\mathrm{S}}\cdot\mathbf{n}_\perp, \quad a_\perp^{\mathrm{F}} := \mathbf{a}^{\mathrm{F}}\cdot\mathbf{n}_\perp. \qquad (95)$$

Hence we have for the speed of the front

$$c^2 = \frac{\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}\,\mu^{\mathrm{S}}. \qquad (96)$$

As $\rho_{22} > 0$, $\mu^{\mathrm{S}} > 0$, the *first condition for hyperbolicity* of the set (88) follows to be

$$a - r(1-a) > 0. \qquad (97)$$

This condition is obviously fulfilled because $r > 0$ and $a$ is not smaller than 1 (see (51)). The speed of propagation (96) describes the shear wave. It is easy to see that in the particular case without the influence of tortuosity $a = 1$ this relation reduces to the classical formula

$$c = c_S = \sqrt{\mu^{\mathrm{S}}/\rho_0^{\mathrm{S}}}. \qquad (98)$$

In this case, according to $(95)_2$, the amplitude in the fluid $a_\perp^{\mathrm{F}}$ is zero, i.e. the shear wave is carried solely by the skeleton.

We proceed to the *longitudinal* part. Now we take the scalar product of relations (94) with the vector $\mathbf{n}$. It follows

$$(\rho_{11}c^2 - (\lambda^{\mathrm{S}} + 2\mu^{\mathrm{S}}))\mathbf{a}^{\mathrm{S}}\cdot\mathbf{n} + (\rho_{12}c^2 - Q)\mathbf{a}^{\mathrm{F}}\cdot\mathbf{n} = 0,$$

$$(\rho_{12}c^2 - Q)\mathbf{a}^{\mathrm{S}}\cdot\mathbf{n} + (\rho_{22}c^2 - \kappa\rho_0^{\mathrm{F}})\mathbf{a}^{\mathrm{F}}\cdot\mathbf{n} = 0, \qquad (99)$$

and the dispersion relation is as follows:

$$r[(1 - r(1-a))c^2 - c_{P1}^2][ac^2 - c_{P2}^2] - \left[r(1-a)c^2 - \frac{Q}{\rho_0^{\mathrm{S}}}\right]^2 = 0, \qquad (100)$$

where

$$c_{P1}^2 := \frac{\lambda^{\mathrm{S}} + 2\mu^{\mathrm{S}}}{\rho_0^{\mathrm{S}}}, \quad c_{P2}^2 := \kappa. \qquad (101)$$

The eigenvalues of this problem have the form

$$c^2 = \frac{1}{2r[a - r(1-a)]}[A \pm \sqrt{B}], \qquad (102)$$

where

$$A := rac_{P1}^2 + [1 - r(1-a)]rc_{P2}^2 - 2\frac{Q}{\rho_0^{\mathrm{S}}}r(1-a),$$

$$B := A^2 - 4r[a - r(1-a)]\left[c_{P1}^2 c_{P2}^2 r - \frac{Q^2}{\rho_0^{\mathrm{S2}}}\right]. \qquad (103)$$

It can be easily shown that under condition (97) $B > 0$ for all $a \geqslant 1$, $Q \geqslant 0$. However, $c^2$ defined by (102) is positive only if the additional condition

$$Q \leqslant \rho_0^{\mathrm{S}}\sqrt{r}c_{P1}c_{P2} \equiv \sqrt{\rho_0^{\mathrm{F}}\kappa(\lambda^{\mathrm{S}} + 2\mu^{\mathrm{S}})}, \qquad (104)$$

on $Q$ is satisfied. This is the *second condition for hyperbolicity*.

If $a = 1$, $Q = 0$ then $c$ is equal to either $c_{P1}$ or $c_{P2}$ which means that the set is unconditionally hyperbolic.

The two solutions for $c^2$ define two longitudinal modes of propagation, $P1$ and $P2$. The $P2$-mode, called the Biot wave or the *slow wave* in the theory of porous materials, is also known as the *second sound* and it appears in all two-component systems described by hyperbolic field equations. For instance, it is known in the theory of binary mixtures of fluids which is applied to describe dynamic properties of liquid helium. For porous materials, it has been discovered by Frenkel in 1944 [5]. After the derivation of the dispersion relation for monochromatic waves—we consider this problem in the next subsection—he wrote: "...for large values of the parameters $\zeta = \mu/\kappa$ ($\equiv \pi$ in our notation) one of the roots corresponds to waves with a very small damping, and the other—to waves with a very large damping. The waves of the second kind are thus really nonexistent". He did not consider these waves any further and they were discovered again by Biot in 1956 [7].

### 4.2. Monochromatic bulk waves, low and high frequency limits, attenuation

The above analysis yields solely the propagation properties of the wave front $\mathscr{S}$. We do not learn anything about, for instance, the attenuation of waves. For this reason we proceed to analyze monochromatic waves. As we will see, the speeds of propagation obtained above follow in the limit of frequency $\omega \to \infty$.

For simplicity we leave out the memory effect in the source of momentum. This simplification affects the behavior of attenuation in the high frequency range but speeds of monochromatic waves change only little. It means that we assume the source to have the form

$$\hat{\mathbf{p}} = \pi(\mathbf{v}^{\mathrm{F}} - \mathbf{v}^{\mathrm{S}}), \qquad (105)$$

where the permeability coefficient $\pi$ is constant.

We seek solutions of Eqs. (88) in the form of the following plain monochromatic waves of given frequency, $\omega$:

$$\mathbf{v}^S = \mathbf{V}^S \mathscr{E}, \quad \mathbf{v}^F = \mathbf{V}^F \mathscr{E}, \quad \mathbf{e}^S = \mathbf{E}^S \mathscr{E}, \quad \varepsilon = E^F \mathscr{E},$$

$$\mathscr{E} := \exp[\mathrm{i}(\mathbf{k} \cdot \mathbf{x} - \omega t)], \tag{106}$$

where $\mathbf{V}^S$, $\mathbf{V}^F$, $\mathbf{E}^S$, $E^F$ are constant complex amplitudes, and $\mathbf{k}$ is the wave vector. This should be understood in the following way: the wave number, $k := \sqrt{\mathbf{k} \cdot \mathbf{k}} = \operatorname{Re} k + i \operatorname{Im} k$, is complex and the direction of propagation $\mathbf{n} := \mathbf{k}/k$ real which yields $\exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}) = \exp(-\operatorname{Im} k(\mathbf{n} \cdot \mathbf{x})) \exp[\mathrm{i}(\operatorname{Re} k(\mathbf{n} \cdot \mathbf{x}) - \omega t)]$, i.e. the function $\mathscr{E}$ splits into contributions of attenuation and progressive wave.

Substitution of this ansatz in the field equations yields the following compatibility conditions:

$$[\rho_{11}\omega^2 \mathbf{1} - \lambda^S \mathbf{k} \otimes \mathbf{k} - \mu^S(k^2 \mathbf{1} + \mathbf{k} \otimes \mathbf{k}) + \mathrm{i}\pi\omega \mathbf{1}]\mathbf{V}^S$$
$$+ [\rho_{12}\omega^2 \mathbf{1} - Q\mathbf{k} \otimes \mathbf{k} - \mathrm{i}\pi\omega \mathbf{1}]\mathbf{V}^F = 0,$$

$$[\rho_{12}\omega^2 \mathbf{1} - Q\mathbf{k} \otimes \mathbf{k} - \mathrm{i}\pi\omega \mathbf{1}]\mathbf{V}^S$$
$$+ [\rho_{22}\omega^2 \mathbf{1} - \kappa\rho_0^F \mathbf{k} \otimes \mathbf{k} + \mathrm{i}\pi\omega \mathbf{1}]\mathbf{V}^F = 0. \tag{107}$$

The problem of existence of such waves reduces, as usual, to the eigenvalue problem with the eigenvector $[\mathbf{V}^S, \mathbf{V}^F]$. As in the case of propagation of fronts, we split the problem into two parts: in the direction $\mathbf{k}_\perp$ perpendicular to $\mathbf{k}$ (transversal modes) and in the direction of the wave vector $\mathbf{k}$ (longitudinal modes).

For transversal modes (monochromatic shear waves) we have

$$[\rho_{11}\omega^2 - \mu^S k^2 + \mathrm{i}\pi\omega]V_\perp^S + [\rho_{12}\omega^2 - \mathrm{i}\pi\omega]V_\perp^F = 0,$$
$$k^2 = \mathbf{k} \cdot \mathbf{k},$$

$$[\rho_{12}\omega^2 - \mathrm{i}\pi\omega]V_\perp^S + [\rho_{22}\omega^2 + \mathrm{i}\pi\omega]V_\perp^F = 0,$$

$$V_\perp^S = \mathbf{V}^S \cdot \mathbf{k}_\perp, \quad V_\perp^F = \mathbf{V}^F \cdot \mathbf{k}_\perp. \tag{108}$$

The dispersion relation can in this case be written in the following form:

$$\omega\left\{(\rho_{11}\rho_{22} - \rho_{12}^2)\left(\frac{\omega}{k}\right)^2 - \mu^S \rho_{22}\right\}$$
$$+ \mathrm{i}\pi\left\{(\rho_{11} + \rho_{22} + 2\rho_{12})\left(\frac{\omega}{k}\right)^2 - \mu^S\right\} = 0, \tag{109}$$

i.e.

$$\left(\frac{\omega}{k}\right)^2 = \frac{\omega r a + \mathrm{i}(\pi/\rho_0^S)}{\omega r[a - r(1-a)] + \mathrm{i}(\pi/\rho_0^S)(1+r)} c_S^2, \quad c_S^2 = \frac{\mu^S}{\rho_0^S}. \tag{110}$$

Consequently, neither the phase speed $\omega/\operatorname{Re} k$ nor the attenuation $\operatorname{Im} k$ of monochromatic shear waves are dependent on the coupling coefficient $Q$.

In the two limits of frequencies we have then the following solutions:

$$\omega \to 0:$$

$$\lim_{\omega \to 0}\left(\frac{\omega}{\operatorname{Re} k}\right)^2 = \frac{\mu^S}{\rho_0^S + \rho_0^F}, \quad \lim_{\omega \to 0}(\operatorname{Im} k) = 0,$$

$$\rho_{11} + \rho_{22} + 2\rho_{12} \equiv \rho_0^S + \rho_0^F,$$

$$\omega \to \infty:$$

$$\lim_{\omega \to \infty}\left(\frac{\omega}{\operatorname{Re} k}\right)^2 = \frac{\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}\mu^S,$$

$$\lim_{\omega \to \infty}(\operatorname{Im} k) = \frac{\pi}{2\sqrt{\rho_0^S \mu^S}}\frac{1}{a^2}\sqrt{\frac{a}{a - r(1-a)}}. \tag{111}$$

The first result checks with results of the classical one-component model commonly used in soil mechanics. The denominator $\rho_0^S + \rho_0^F$ indicates that both components move in phase. This gives rise to the lower speed of propagation than in the limit of infinite frequencies. The speed in the second one is identical with this of formula (96). Hence the propagation of the front of shear waves is identical with the propagation of monochromatic waves of infinite frequency. Notice that the attenuation in this limit is finite.

We demonstrate further properties of these monochromatic waves on a numerical example.

In the direction $\mathbf{k}$, for longitudinal modes, we obtain the dispersion relation

$$[\rho_{11}\omega^2 - (\lambda^S + 2\mu^S)k^2 + \mathrm{i}\pi\omega][\rho_{22}\omega^2 - \kappa\rho_0^F k^2 + \mathrm{i}\pi\omega]$$
$$- (\rho_{12}\omega^2 - Qk^2 - \mathrm{i}\pi\omega)^2 = 0. \tag{112}$$

It is convenient to write this relation in the following form:

$$\omega\left\{[1 - r(1-a)]\left(\frac{\omega}{k}\right)^2 - c_{P1}^2\right\}\left\{a\left(\frac{\omega}{k}\right)^2 - c_{P2}^2\right\}$$
$$+ \frac{1}{r}\mathrm{i}\frac{\pi}{\rho_0^S}\left(\frac{\omega}{k}\right)^2\left\{(1+r)\left(\frac{\omega}{k}\right)^2 - rc_{P2}^2 - c_{P1}^2 - 2\frac{Q}{\rho_0^S}\right\}$$
$$- \frac{1}{r}\omega\left\{r(1-a)\left(\frac{\omega}{k}\right)^2 - \frac{Q}{\rho_0^S}\right\}^2 = 0. \tag{113}$$

Let us check again the two limits of frequencies: $\omega \to 0$ and $\infty$. In the first case the first contribution in (113) identically vanishes and we obtain

$$\omega \to 0:$$

$$c_0 := \lim_{\omega \to 0}\left(\frac{\omega}{\operatorname{Re} k}\right),$$

$$c_0^2\left\{(1+r)c_0^2 - rc_{P2}^2 - c_{P1}^2 - 2\frac{Q}{\rho_0^S}\right\} = 0, \quad \lim_{\omega \to 0}(\operatorname{Im} k) = 0. \tag{114}$$

Obviously, we obtain two real solutions of this equation

$$\lim_{\omega \to 0} \left( \frac{\omega}{\operatorname{Re} k} \right)^2 \bigg|_1 := C_{0P1}^2 = \frac{c_{p1}^2 + rc_{p2}^2 + 2(Q/\rho_0^{\mathrm{S}})}{1 + r}$$

$$\equiv \frac{\lambda^{\mathrm{S}} + 2\mu^{\mathrm{S}} + \rho_0^{\mathrm{F}}\kappa + 2Q}{\rho_0^{\mathrm{S}} + \rho_0^{\mathrm{F}}},$$

$$\lim_{\omega \to 0} \left( \frac{\omega}{\operatorname{Re} k} \right)^2 \bigg|_2 := c_{0P2}^2 = 0. \tag{115}$$

These are squares of the speeds of propagation of two longitudinal modes in the limit of zero frequency. Clearly, the second mode, $P2$-wave, does not propagate in this limit. Both limits are independent of the tortuosity coefficient, $a$. Result (115) checks with the relation for the speed of longitudinal waves used in the classical one-component model of soil mechanics provided $Q = 0$. Again the components move in phase.

In the second limit case we have

$$\omega \to \infty:$$

$$c_\infty := \lim_{\omega \to \infty} \left( \frac{\omega}{\operatorname{Re} k} \right),$$

$$r\{[1 - r(1 - a)]c_\infty^2 - c_{P1}^2\}\{ac_\infty^2 - c_{P2}^2\}$$
$$- \left\{ r(1 - a)c_\infty^2 - \frac{Q}{\rho_0^{\mathrm{S}}} \right\}^2 = 0. \tag{116}$$

This coincides with relation (100). Consequently, the limit $\omega \to \infty$ gives indeed the properties of the front of acoustic longitudinal waves in the system.

Simultaneously we obtain the following attenuation in the limit of infinite frequencies:

$$\lim_{\omega \to \infty} (\operatorname{Im} k) = \frac{\pi \Gamma_1}{2\rho_0^{\mathrm{S}} r \Gamma_2},$$

$$\Gamma_1 = c_\infty \left[ 1 + r - \frac{1}{c_\infty^2} \left( c_{P1}^2 + rc_{P2}^2 + 2\frac{Q}{\rho_0^{\mathrm{S}}} \right) \right],$$

$$\Gamma_2 = c_{P1}^2 \left( a - \frac{c_{P2}^2}{c_\infty^2} \right) + c_{P2}^2 \left( 1 - r(1 - a) - \frac{c_{P1}^2}{c_\infty^2} \right)$$
$$+ 2\frac{Q}{\rho_0^{\mathrm{S}}} \left( 1 - a - \frac{Q}{r\rho_0^{\mathrm{S}}c_\infty^2} \right). \tag{117}$$

Hence both limits of attenuation for the $P1$-wave and the $P2$-wave are finite.

Let us mention that in many works on wave analysis the attenuation of waves is measured not by the imaginary part of the wave number $\operatorname{Im} k$ but by the so-called quality factor $Q$ (unfortunately, the notation is the same as for the Biot coupling parameter!) which is proportional to $\omega/\operatorname{Im} k$. Consequently, for a finite limit of $\operatorname{Im} k$ in $\omega \to \infty$ the quality factor in this limit becomes infinite (i.e. $\lim_{\omega \to \infty} 1/Q = 0$). For this reason, it is sometimes claimed that waves are not attenuated in this limit. Physically, this statement does not make sense as the dissipation in the system, responsible for the attenuation, is different from zero for any monochromatic wave.

We proceed with the presentation of a numerical result in the whole range of frequencies $\omega \in [0, \infty)$ (e.g. compare [21]). We use the following numerical data:

$$c_{P1} = 2500 \, \mathrm{m/s}, \quad c_{P2} = 1000 \, \mathrm{m/s}, \quad c_S = 1500 \, \mathrm{m/s},$$

$$\rho_0^{\mathrm{S}} = 2500 \, \mathrm{kg/m^3}, \quad r = 0.1, \quad \pi = 10^8 \, \mathrm{kg/m^3 s},$$

$$Q = 0.8 \, \mathrm{GPa}, \quad n_0 = 0.4, \quad a = 1.75. \tag{118}$$

Speeds $c_{P1}$, $c_{P2}$, $c_S$, mass density $\rho_0^{\mathrm{S}}$ (i.e. $\rho_0^{\mathrm{SR}} = 4167 \, \mathrm{kg/m^3}$ for the porosity $n_0 = 0.4$) and fraction $r = \rho_0^{\mathrm{F}}/\rho_0^{\mathrm{S}}$ possess values typical for many granular materials under a confining pressure of a few atmospheres and saturated by water. In units that are standard for soil
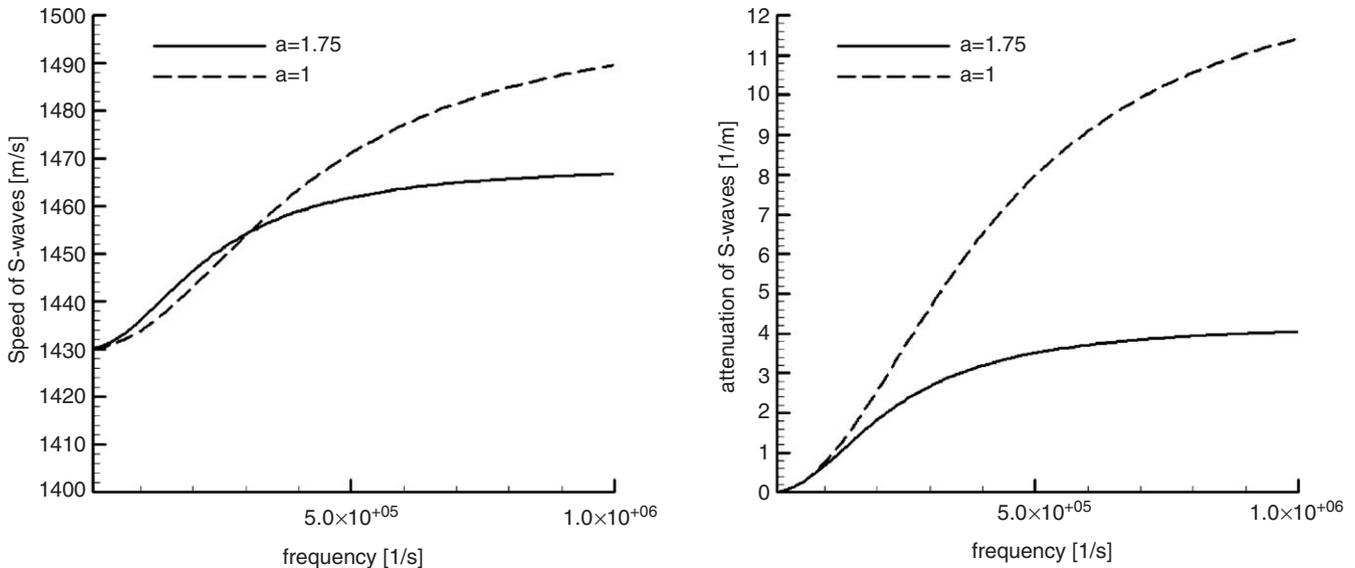


Fig. 3. Speed of propagation and attenuation of monochromatic $S$-waves for two values of the tortuosity coefficient $a$ : 1.75 (Biot), 1.00 (simple mixture) [21].

mechanics the permeability $\pi$ corresponds to app. 0.1 Darcy. The coupling coefficient $Q$ has been estimated by means of the Gassmann relation. The tortuosity coefficient $a = 1.75$ follows from the Berryman formula $(89)_5$.

Transversal waves described by relation (110) are characterized by the distribution of speeds and attenuation in function of frequency as shown in Fig. 3. The solid lines correspond to the solution of Biot's model and the dashed lines to the solution of the simple mixture model.

It is clear that the qualitative behavior of the speed of propagation is the same in both models. In the range of high frequencies, it is a few percent smaller in Biot's model than in the simple mixture model. A large quantitative difference between these models appears for the attenua-

tion. In the range of higher frequencies it is much smaller in Biot's model, i.e. tortuosity described by the added mass $\rho_{12}$ lowers the dissipation of shear waves.

The latter property is illustrated in Fig. 4 where we plot the attenuation of the front of shear waves, i.e. $\lim \omega \to \infty \, \mathrm{Im} \, k$, as a function of the tortuosity coefficient, $a$. This behavior of the attenuation indicates that the damping of waves created by the tortuosity must be placed in the model somewhere else. As we already indicated, Biot did this by making the diffusion coefficient, $\pi$, dependent on the frequency. The influence of added mass plays indeed an important role in the theory of suspensions but, according to the above result, it seems to be rather doubtful if it should be included in the theory of porous materials. We proceed to longitudinal waves. The solid lines in the following figures correspond again to Biot's model, the dashed lines to the simple mixture model. In order to show separately the influence of the tortuosity, $a$, and of the coupling, $Q$, we plot as well the solutions with $a = 1$ (dashed–dotted lines) and the solutions with $Q = 0$ (dashed double dotted lines).

Even though similar again the quantitative differences are much more substantial for $P1$-waves (Fig. 5). This is primarily the influence of the coupling through partial stresses described by the parameter $Q$. The simple mixture model ($Q = 0, a = 1$) as well as Biot's model with $Q = 0$ yield speeds of these waves which differ only a few percent from one another (lower curves in the left diagram). The coupling, $Q$, shifts the curves to higher values and reduces the difference caused by the tortuosity. This result does not seem to be very realistic because the real differences between low frequency and high frequency speeds were measured in soils to be rather as big as indicated by the simple mixture model. This may be an indication that values of the coupling parameter, $Q$, given by Gassmann's relations, are too big for real granular materials.
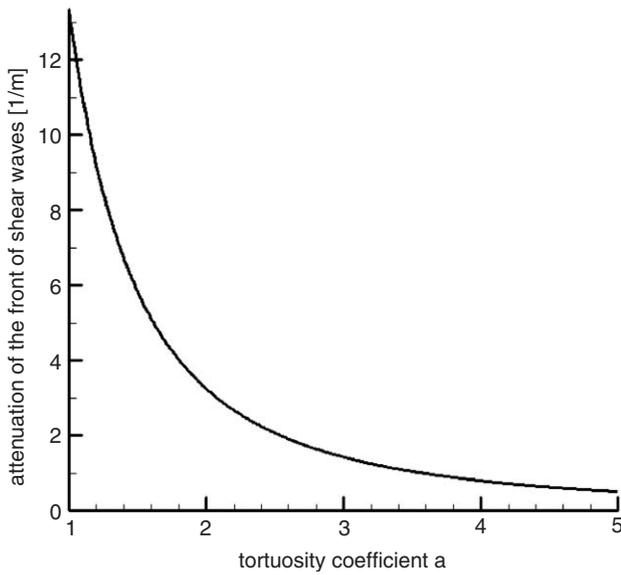


Fig. 4. Attenuation of the front of shear waves in function of the tortuosity coefficient $a$ [21].
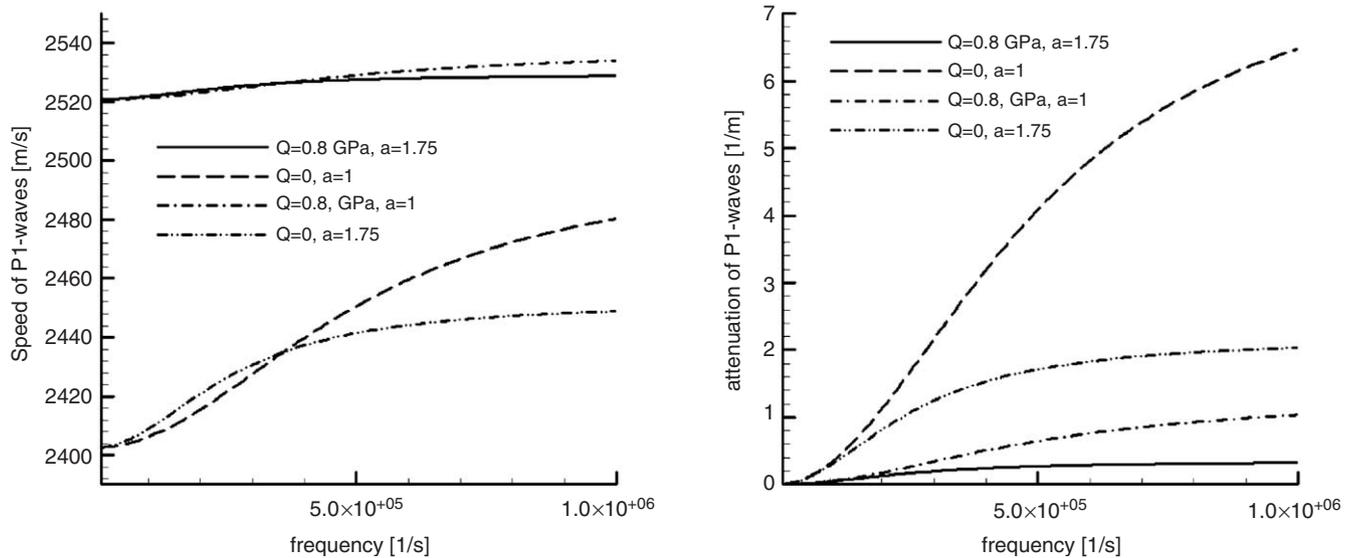


Fig. 5. Speed of propagation and attenuation of monochromatic $P1$-waves for various coupling parameters $Q$ and tortuosity coefficients $a$ [21].
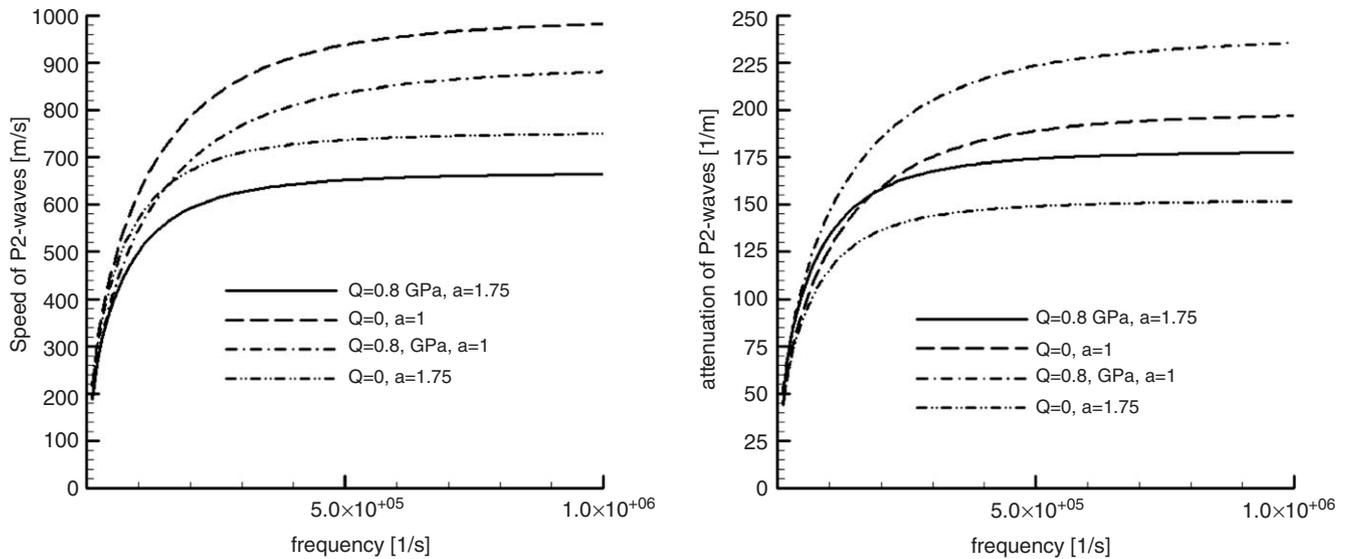
Fig. 6. Speed of propagation and attenuation of monochromatic *P*2-waves for various coupling parameters *Q* and tortuosity coefficients *a* [21].

Both the tortuosity, *a*, and the coupling, *Q*, reduce the attenuation quite considerably as indicated in the right panel of Fig. 5.

In spite of some claims in the literature to the contrary, the tortuosity *a* does not influence the existence of the slow (*P*2-) wave (Fig. 6). Speeds of this wave are again qualitatively similar in Biot's model and in the simple mixture model. The maximum differences appear in the range of high frequencies and reach 35%. The same concerns the attenuation even though quantitative differences are not so big (app. 8%).

The above examples yield the following inferences:

(1°) We have demonstrated on the example of acoustic waves that the tortuosity coefficient, *a*, and the coupling parameter, *Q*, have a quantitative but not a qualitative influence on speeds and attenuations of waves. Comparison of results for Biot's model with these for the simple mixture model in which the tortuosity $a = 1$ and the coupling parameter $Q = 0$ shows that both models are hyperbolic provided the parameter *Q* satisfies a condition bounding this parameter from above. In particular, both models predict the existence of the *P*2-wave. Speeds and attenuations of monochromatic *P*1-, *P*2- and *S*-waves are qualitatively the same but there are quantitative discrepancies.

(2°) Tortuosity introduced to the model through the relative acceleration yields dissipation solely due to the modification of the relative motion. Namely, if we assume the permeability coefficient $\pi = 0$ the dissipation in isothermal processes without relaxation of porosity vanishes but an influence of relative accelerations remains in the model. This is due to the fact that the tortuosity, in contrast to the porosity, is not introduced as a field described by its own field equation. This is an explanation of a rather unexpected behavior of the attenuation of

monochromatic waves. Inspection of the figures shown in this section makes clear that the presence of the tortuosity $a \neq 1$ yields a smaller rather than a bigger attenuation as it would be for a dissipative field. This may be explained by the fact that the tortuosity reduces the relative velocity $\mathbf{v}^F - \mathbf{v}^S$ and, consequently, it reduces the contribution to dissipation $\pi(\mathbf{v}^F - \mathbf{v}^S) \cdot (\mathbf{v}^F - \mathbf{v}^S)$. This unphysical behavior has not been indicated in the literature because in the investigation of monochromatic waves it is covered by an influence of the viscous effects included in the variable permeability coefficient $\pi(\omega)$.

(3°) We have demonstrated that a rather moderate value of the parameter *Q* suggested by the classical Gassmann relation for granular materials leads to an unreasonable increment of speeds of propagation and reduction of attenuation. In addition, the speed of propagation of monochromatic *P*1-waves becomes very flat as a function of frequency. This contradicts observations in soil mechanics and geotechnics and indicates that the Gassmann relation predicts too big values of this parameter.

## 5. Boundary conditions and surface waves

Geotechnical and seismic applications of surface waves demonstrate the practical importance of this branch of acoustics. These waves enable cheap nondestructive testing of many complex materials and yield simple engineering methods of field investigation of soils, ground water or searching for landmines and other obstacles. In particular, Rayleigh waves in heterogeneous materials play an important role. An extensive presentation of these methods can be found in the book [52].

The analysis of surface waves in saturated porous materials shows that poroelasticity predicts a much richer structure of surface waves than classical elasticity which is

the theoretical tool for classical surface waves used in geotechnics. Possibilities related to the multicomponent character of porous materials and different structures of boundaries can be found in the review article [53] and in the above quoted book (the contributions of Wilmanski, pp. 203–276, and Albers, pp. 277–324). Simultaneously, the analytical complexity of the problem yields the limitation of theoretical investigations to homogeneous materials which leaves out important modes of propagation and essential fields of application. In this section, we present only a very brief presentation of such waves.

## 5.1. Boundary conditions for poroelastic materials

The structure of field equations for poroelastic materials indicates that we need two vectorial boundary conditions. Namely, we have two equations following from the momentum balance and one scalar equation for changes of porosity. However, the latter has the general form of an evolution equation and requires only an initial condition.

One of the vectorial boundary conditions is obvious, and it is identical with the condition appearing in the classical theory of elasticity. Due to the continuity of the stress vector on the material surface we can write approximately

$$(\mathbf{T}^S \mathbf{n} - p^F \mathbf{n})|_{\partial \mathscr{B}_0} = \mathbf{t}_{\text{ext}}, \tag{119}$$

where $\mathbf{n}$ denotes the unit vector normal to the boundary $\partial \mathscr{B}_0$, $\mathbf{t}_{\text{ext}}$ is a given density of the external loading. In this relation, the approximation is connected with the relative motion and it is twofold: in the bulk stress an influence of diffusion is neglected and, for the bulk motion, it is assumed that the material surface of the skeleton (true boundary) and of the bulk medium coincide. The order of magnitude of diffusion contributions (mass density × square of the diffusion velocity) is of the order of a few kilopascals and can be neglected under normal conditions in comparison with components of stresses. It should be stressed that one cannot prescribe such a condition to each component separately. It has been shown by Terzaghi in his famous Gedankenexperiment ([2], see also [54]) that the distribution of the external load $\mathbf{t}_{\text{ext}}$ between the components changes in time: at the initial instant the full loading is taken over by the fluid component and then it is divided between components in a time-dependent manner until for the infinite time the whole external loading is carried by the skeleton.

The second vectorial boundary condition is characteristic for the permeable boundary of the porous material. If this is the boundary with a fluid outside which flows out of the porous material then two conditions must be specified. The first one concerns the transport of mass of the fluid through the boundary. This is assumed to be continuous (i.e. there is no sink of the mass on the boundary) and the amount of the fluid which flows per unit surface and in unit time is given by a *driving force*. The latter is assumed to be equal to the difference between the external pressure and the pore pressure. Such a condition—an interfacial

counterpart of Darcy's law—has been proposed by Deresiewicz (e.g. [55]) in his research of surface waves. It has the form

$$\rho_0^F (\mathbf{v}^F - \mathbf{v}^S) \cdot \mathbf{n} - \alpha(p^F - np_{\text{ext}})|_{\partial \mathscr{B}_0} = 0. \tag{120}$$

The material parameter, $\alpha$ (sometimes denoted by $1/T$, e.g. [56,57,53]), is the *surface permeability coefficient*, and it reflects the existence of a boundary layer in the porous material which is replaced by the condition on the interface. It is an overall macroscopic description of this layer which is created by the flow of the fluid component from conditions specified by the porous material (i.e. by the permeability $\pi$, porosity $n$, a geometry of the microscopic vicinity of the boundary such as the shape of openings of channels, their average orientation with respect to the surface normal, etc.) to the free space of a pure fluid. It is clear that the limit $\alpha \to 0$ corresponds to the impermeable (sealed) boundary, and the limit $\alpha \to \infty$ corresponds to the continuity of pressure in the fluid: $p^F = n_0 p_{\text{ext}}$. Such a boundary condition is used, for instance, in theories of porous materials with a rigid skeleton which are applied in the description of various geotechnical diffusion and seepage processes.

Considerations concerning the derivation of such a condition on an interface between two porous materials can be found in the work of Gurevich and Schoenberg [58].

We still need two scalar conditions to obtain a well-posed boundary value problem. These conditions concern the tangential components of both velocities. For an ideal fluid, we do not have any boundary layers related to the tangential motion and, consequently, these components must be equal

$$(\mathbf{v}^F - \mathbf{v}^S) - (\mathbf{v}^F \cdot \mathbf{n} - \mathbf{v}^S \cdot \mathbf{n})\mathbf{n}|_{\partial \mathscr{B}_0} = 0. \tag{121}$$

This is not true if the fluid is viscous. Such a case was considered by Beavers and Joseph [59] (see also [60,61]) who proposed the analogue of the Deresiewicz condition for tangential components. We do not need this condition in this work.

## 5.2. Surface waves—simple mixture model

The beginning of the research of propagation of surface waves in porous materials goes back to the works of Deresiewicz (e.g. [55], for further references see [52]). Within Biot's model he considered Rayleigh waves on the interface porous material/fluid. This wave is similar to the Rayleigh wave of classical elasticity but it leaks its energy into the P2-wave. This analysis was performed in the full frequency range. The analysis of Feng and Johnson [56,57] for the high frequency limit revealed the existence of an additional surface mode, an analogue of the Stoneley wave, which is slower than the P2-wave. In addition, there exists an additional leaky mode which is the Stoneley wave in the outside fluid. The existence of the true surface mode is strongly dependent on the surface permeability. A brief review of surface modes of propagation and experimental

methods for their investigation can be found in the paper of Nagy [53].

In this section, we present very briefly the theoretical construction of surface waves on the basis of the simple mixture model [62]. It means that we assume $Q = 0$ and $\rho_{12} = 0$ (i.e. $a = 1$) in Biot's model. In addition, we assume that the bulk permeability $\pi$ is constant. In contrast to these simplifications, we show that the results match qualitatively very well available results for Biot's model. Simultaneously, calculations for the simple mixture model are much simpler and can be easily made in the whole frequency range (compare the contribution of Albers in [52] as well as [63,64]).

We use displacements for both components, $\mathbf{u}^S$, $\mathbf{u}^F$. Then Eqs. (88) and (89) can be replaced by the set

$$\frac{\partial^2 \mathbf{u}^S}{\partial t^2} = (1 - 2c_s^2)\text{grad div }\mathbf{u}^S + 2c_s^2 \text{div grad }\mathbf{u}^S + \pi \frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S),$$

$$\frac{\partial^2 \mathbf{u}^F}{\partial t^2} = c_f^2 \text{grad div }\mathbf{u}^F - \pi \frac{\partial}{\partial t}(\mathbf{u}^F - \mathbf{u}^S), \tag{122}$$

where we use the following dimensionless notation:

$$t \to \frac{t}{\tau}, \quad \mathbf{x} \to \frac{\mathbf{x}}{c_{P1}\tau}, \quad \mathbf{u}^S \to \frac{\mathbf{u}^S}{c_{P1}\tau}, \quad \mathbf{u}^F \to \frac{\mathbf{u}^F}{c_{P1}\tau}, \quad \pi \to \frac{\pi\tau}{\rho_0^S},$$

$$c_s = \frac{c_S}{c_{P1}}, \quad c_s = \frac{c_{P2}}{c_{P1}}, \quad r = \frac{\rho_0^F}{\rho_0^S} \tag{123}$$

and definitions (98) and (101) are applied in order to eliminate material parameters. The constant $\tau$ normalizing time is arbitrary. It may be, for instance, identified with the inverse of one of the characteristic frequencies of the porous medium $\pi/2\rho_0^F$ or $\pi/2\rho_0^S$.

We seek the solution by means of the following scalar and vector potentials:

$$\mathbf{u}^S = \text{grad }\varphi^S + \text{rot }\psi^S, \quad \mathbf{u}^F = \text{grad }\varphi^F + \text{rot }\psi^F. \tag{124}$$

We consider surface waves on the plain boundary of a semi-infinite porous medium. For such waves, we introduce Cartesian coordinates with the $z$-axis perpendicular to the boundary and oriented into the porous medium. The boundary coincides with $z = 0$. Then the following ansatz for the two-dimensional problem of propagation in the direction of the $x$-axis is appropriate

$$\varphi^S = A^S(z)\exp[i(kx - \omega t)], \quad \varphi^F = A^F(z)\exp[i(kx - \omega t)],$$

$$\psi_y^S = B^S(z)\exp[i(kx - \omega t)], \quad \psi_y^F = B^F(z)\exp[i(kx - \omega t)],$$

$$\psi_x^S = \psi_z^S = \psi_x^F = \psi_z^F = 0. \tag{125}$$

Substitution into the field equations (122) yields the following compatibility conditions:

$$\left[c_f^2\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right) + \omega^2\right]A^F + i\frac{\pi}{r}\omega(A^F - A^S) = 0,$$

$$\left[\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right) + \omega^2\right]A^S - i\pi\omega(A^F - A^S) = 0,$$

$$\left[c_s^2\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - k^2\right) + \omega^2 + \frac{i\pi\omega}{\omega + i(\pi/r)}\right]B^S = 0. \tag{126}$$

These ordinary differential equations can be easily solved. Let us mention that this would not be the case for heterogeneous materials for which material parameters are functions of the position. This important practical problem is solved approximately for classical Rayleigh waves [52]. For porous materials the problem has not been formulated as yet.

We write the solution of the set in the form

$$A^F = A_f^1 e^{\gamma_1 z} + A_f^2 e^{\gamma_2 z}, \quad A^S = A_s^1 e^{\gamma_1 z} + A_s^2 e^{\gamma_2 z}, \quad B^S = B_s e^{\zeta z}, \tag{127}$$

where the exponents follow from (126) in the form

$$\left(\frac{\zeta}{k}\right)^2 = 1 - \frac{1}{c_s^2}\left(1 + \frac{i\pi}{\omega + i(\pi/r)}\right)\left(\frac{\omega}{k}\right)^2, \tag{128}$$

and

$$c_f^2\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]^2 + \left[1 + \left(1 + \frac{1}{r}\right)\frac{i\pi}{\omega}\right]\left(\frac{\omega}{k}\right)^4$$
$$+ \left[1 + c_f^2 + \left(c_f^2 + \frac{1}{r}\right)\frac{i\pi}{\omega}\right]\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]\left(\frac{\omega}{k}\right)^2 = 0. \tag{129}$$

These are equations for $\zeta$ and $\gamma$. For the existence of surface waves, we have to require that the exponents possess negative real parts. Simultaneously, the coefficients in (127) are related in the following way:

$$A_f^1 = \delta_f A_s^1, \quad A_s^2 = \delta_s A_f^2, \tag{130}$$

$$\delta_f := \frac{1}{r}\frac{(i\pi/\omega)(\omega^2/k^2)}{c_f^2[(\gamma_1/k)^2 - 1] + (\omega/k)^2 + (i\pi/\omega r)(\omega^2/k^2)}, \tag{131}$$

$$\delta_s := \frac{(i\pi/\omega)(\omega^2/k^2)}{[(\gamma_2/k)^2 - 1] + (\omega/k)^2 + (i\pi/\omega)(\omega^2/k^2)}. \tag{132}$$

In order to see the difference in the behavior of the exponents in low and high frequency limits, we solve Eqs. (128) and (129) in these limits.

In the high frequency approximation we obtain

$$\frac{1}{\omega} \ll 1: \quad \left(\frac{\zeta}{k}\right)^2 = 1 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2,$$

$$\left(\frac{\gamma_1}{k}\right)^2 = 1 - \left(\frac{\omega}{k}\right)^2, \quad \left(\frac{\gamma_2}{k}\right)^2 = 1 - \frac{1}{c_f^2}\left(\frac{\omega}{k}\right)^2, \tag{133}$$

and

$$\delta_f = \delta_s = 0. \tag{134}$$

On the other hand, the limit $\omega \to 0$ is singular. We solve the problem by the method of singular perturbations.

It follows:

$$\omega \ll 1: \quad \left(\frac{\zeta}{k}\right)^2 = 1 - \frac{r+1}{c_s^2}\left(\frac{\omega}{k}\right)^2,$$

$$\left(\frac{\gamma_1}{k}\right)^2 = 1 - \frac{r+1}{rc_f^2 + 1}\left(\frac{\omega}{k}\right)^2,$$

$$\left(\frac{\gamma_2}{k}\right)^2 = 1 - \frac{rc_f^4 + 1}{c_f^2(rc_f^2 + 1)}\left(\frac{\omega}{k}\right)^2 - \frac{i\pi}{\omega}\frac{rc_f^2 + 1}{rc_f^2}\left(\frac{\omega}{k}\right)^2, \quad (135)$$

and for the coefficients of the amplitudes

$$\delta_f = 1 - \frac{\omega r}{i\pi}\frac{1 - c_f^2}{1 + rc_f^2}, \quad \delta_s = -rc_f^2\left(1 - \frac{\omega r}{i\pi}\frac{1 - c_f^2}{1 + rc_f^2}\right). \quad (136)$$

Clearly, the relation for $\gamma_2/k$ contains a singularity.

The solution for exponents (127) still leaves three unknown constants $B_s$, $A_f^2$, $A_s^1$ which must be specified from boundary conditions.

For the contact of the porous materials with vacuum through a sealed boundary ($\alpha = 0$) the boundary conditions (119), (120) have, in dimensionless notation, the following form:

$$\frac{1}{\rho_0^S c_{P1}^2} T_{13}\bigg|_{z=0} \equiv \frac{1}{\rho_0^S c_{P1}^2} T_{13}^S\bigg|_{z=0} = c_s^2\left(\frac{\partial u_1^S}{\partial z} + \frac{\partial u_3^S}{\partial x}\right)\bigg|_{z=0} = 0, \quad (137)$$

$$\frac{1}{\rho_0^S c_{P1}^2} T_{33}\bigg|_{z=0}$$
$$\equiv \frac{1}{\rho_0^S c_{P1}^2}(T_{33}^S - p^F)\bigg|_{z=0}$$
$$= \left(\frac{\partial u_1^S}{\partial x} + \frac{\partial u_3^S}{\partial z}\right) - 2c_s^2\frac{\partial u_1^S}{\partial x} - c_f^2 r\left(\frac{\rho^F}{\rho_0^F} - 1\right)\bigg|_{z=0} = 0, \quad (138)$$

$$\frac{\partial}{\partial t}(u_3^F - u_3^S)\bigg|_{z=0} = 0, \quad (139)$$

where $T_{13}$, $T_{13}^S$, $T_{33}$, $T_{33}^S$ are components of stress tensors and the first two conditions mean that the surface $z = 0$ is stress-free (far-field approximation), and the last condition means that there is no transport of fluid mass through this surface (impermeable boundary). $u_1^S$, $u_3^S$ denote the components of the displacement $\mathbf{u}^S$ in the direction of the $x$-axis and the $z$-axis, respectively, while $u_3^F$ is the $z$-component of the displacement $\mathbf{u}^F$.

We present the solution of this boundary value problem only in the above mentioned two limits of the frequency. The numerical results for the whole spectrum can be found in [63]. As the boundary conditions yield a homogeneous set of algebraic relations we have to require that the determinant is equal to zero. This yields the *dispersion relation* which can be reduced to the following relations for the speeds of propagation of surface waves.

For high frequencies $1/\omega \ll 1$ we have $\delta_s = \delta_f = 0$, and the dispersion relation follows in the form:

$$\mathcal{P}_R\sqrt{1 - \frac{1}{c_f^2}\left(\frac{\omega}{k}\right)^2} + \frac{r}{c_s^4}\left(\frac{\omega}{k}\right)^4\sqrt{1 - \left(\frac{\omega}{k}\right)^2} = 0, \quad (140)$$

where

$$\mathcal{P}_R := \left(2 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2\right)^2 - 4\sqrt{1 - \left(\frac{\omega}{k}\right)^2}\sqrt{1 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2}. \quad (141)$$

Hence for $r = 0$ relation (140) reduces to $\mathcal{P}_R = 0$ which is the Rayleigh dispersion relation for single component elastic continua. In general, Eq. (140) possesses two roots defining two surface waves: a true Stoneley wave which propagates with finite attenuation and with a velocity a bit smaller than $c_f$ as well as a generalized Rayleigh wave which is leaky (i.e. it radiates energy to the $P2$-wave) and propagates with the velocity $c_R$: $c_f < c_R < c_s$. The Rayleigh wave is leaky because its attenuation is unbounded, i.e. such a wave cannot exist in the high-frequency range. Immediately after the initiation, it transforms into bulk waves.

These results are not very surprising because the dispersion relation (140) is identical with the dispersion relation for the so-called Stoneley–Scholte wave (e.g. see: [65]). The only difference is that the real Stoneley–Scholte wave propagates on both sides of the interface and the above presented wave propagates solely below the boundary ($z > 0$) in the porous medium.

Due to the singularity in the limit $\omega \to 0$ we have to perform a singular perturbation again. The expansion with respect to $\sqrt{\omega}$ yields the identity in the zeroth order and the following relation for the higher order:

$$\left(\frac{\omega}{k}\right)\left\{\left(2 - \frac{r+1}{c_s^2}\left(\frac{\omega}{k}\right)^2\right)^2\right.$$
$$\left. - 4\sqrt{1 - \frac{r+1}{c_s^2}\left(\frac{\omega}{k}\right)^2}\sqrt{1 - \frac{r+1}{rc_f^2 + 1}\left(\frac{\omega}{k}\right)^2}\right\}$$
$$+ O(\sqrt{\omega}) = 0. \quad (142)$$

Clearly we obtain two solutions:
(1) A Rayleigh wave of which the velocity is different from zero in the limit $\omega \to 0$ and whose attenuation is of the order $O(\sqrt{\omega})$. The relation for the velocity is similar to relation (141) with the velocities of bulk waves replaced by the low frequency limits

$$c_{0P1}^2 = \frac{\lambda^S + 2\mu^S}{\rho_0^S + \rho_0^F}, \quad c_{0S}^2 = \frac{\mu^S}{\rho_0^S + \rho_0^F}. \quad (143)$$

Particularly we have

$$\frac{r+1}{c_s^2} = c_{P1}^2\frac{\rho_0^S + \rho_0^F}{\mu^S} \equiv \frac{c_{P1}^2}{c_{0S}^2},$$
$$\frac{r+1}{rc_f^2 + 1} = c_{P1}^2\frac{\rho_0^S + \rho_0^F}{\lambda^S + 2\mu^S + \rho_0^F\kappa} \equiv \frac{c_{P1}^2}{c_{0P1}^2}. \quad (144)$$

Consequently,

$$\left(2 - \frac{c_{P1}^2}{c_{0S}^2}\left(\frac{\omega}{k}\right)^2\right)^2 - 4\sqrt{1 - \frac{c_{P1}^2}{c_{0S}^2}\left(\frac{\omega}{k}\right)^2}\sqrt{1 - \frac{c_{P1}^2}{c_{0P1}^2}\left(\frac{\omega}{k}\right)^2} = 0.$$

(145)

(2) The Stoneley wave has a velocity of propagation of the order $O(\sqrt{\omega})$. Hence, it goes to zero in the same way as the velocity of propagation of the $P2$-wave.

For a permeable boundary neither condition (137) nor condition (139) would hold. The first condition (137) would have to possess the right-hand side reflecting the external pressure $p_{ext}$ appearing in the fluid outside the porous material. This change would appear as well in for an impermeable boundary when we did not have the vacuum outside.
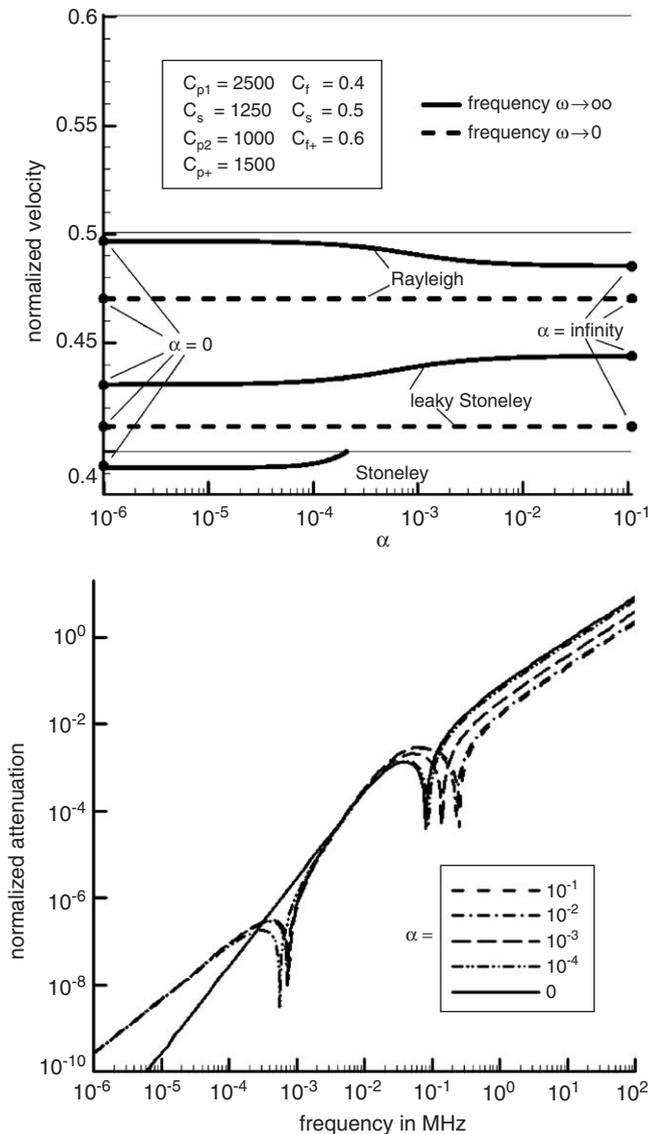


Fig. 7. Some results of Albers for surface waves on the interface porous medium/fluid. Upper panel: velocity of surface waves as a function of surface permeability $\alpha$ for limit frequencies $\omega \to \infty$ and 0. Lower panel: attenuation of leaky Stoneley wave as a function of frequency for different surface permeabilities $\alpha$ [52].

On the other hand, condition (139) must be replaced by the Deresiewicz condition which in the present case has the form

$$\rho_0^F \frac{\partial}{\partial t}(u_3^F - u_3^S) - \alpha(p^F - n_0 p_{ext})\bigg|_{z=0} = 0,$$

(146)

where $p_{ext}$ is an external pressure.

In addition, for a permeable boundary we have to account for the continuity of the mass flux through the boundary. This additional boundary condition is necessary with respect to the existence of an additional constant in the solution for the exterior (in the range $z < 0$).

We omit theoretical considerations for this case and refer to [52] and the forthcoming paper by Albers [64]. The results presented in these works support the view that the simplification made at the beginning of this section gives rise to excellent qualitative results which, as far as comparable, check with earlier results of Feng, Johnson and others obtained for the frequency limit $\omega \to \infty$. As an example, we present in Fig. 7 (upper panel) the relation between velocities of propagation of three surface modes on the permeable boundary as a function of the surface permeability coefficient $\alpha$ as they follow within the simple mixture model in the limits $\omega \to \infty$ and 0. These results of Albers check in all details with the results presented by Nagy [52, Fig. 23]. As we see the true surface mode—Stoneley wave—exists only in the range of the relatively closed boundary. For the open boundary only leaky waves may appear. These results are also confirmed in the new work of Gubaidullin et al. [65] on surface waves described by Biot's model in the full spectrum of frequencies and the two limit values of $\alpha$.

Another result of Albers which seems to be quite new is shown in the lower panel of Fig. 6. Namely, the attenuation of the leaky Stoneley wave indicates the existence of resonances at characteristic frequencies $\pi/2\rho_0^S$ and $\pi/2\rho_0^F$ which are observed experimentally (comp. Fig. 4.13 in [40]) and which are not reported in the work [66].

The above reported results must be still considered as preliminary. The results for the simple mixture model seem to agree quite well with the results for the full Biot model but the problem of influence of the tortuosity remains open. One should incorporate in the analysis the frequency-dependent permeability $\pi(\omega)$ which has not been done yet.

## 6. Final remarks

The above presented review of results concerning the structure of Biot's model of porous materials shows clearly that this model, in spite of its flaws, describes in the linear way processes in poroelastic saturated materials in agreement with fundamental requirements of continuum thermodynamics. The violation of some principles, like thermodynamic admissibility or material objectivity, are characteristic for many linear models including the classical theory of elasticity and are not essential for results

obtained in inertial reference systems. However, it shows as well that extensions to nonlinear processes are not straightforward and should be made by means of appropriate methods of continuum thermodynamics which require, for instance, a constitutive dependence on higher gradients and a proper definition of objective relative accelerations. The problem of extension of the momentum source responsible for tortuosity effects seems to be simple as well once we have given this contribution in the linear model in a hereditary form characteristic for viscous effects.

In addition, many discussions and controversies in this field of research seem to be related rather to the interpretation of field quantities, in particular to their microstructural interpretation, than to the physical contents of the model. Therefore, it would be useful to perform the averaging of microstructural properties in a more realistic situation than as proposed by Biot and Willis and presented in Section 3.4.

It seems to be also clear that it is a waste of effort to try to construct a true variational principle as the Biot model contains a nonequilibrium variable, the increment of fluid contents which rules out the existence of such a principle.

Finally, it seems to be obvious that the most important effects observed by the propagation of waves are described not only by Biot's model but even by its simplified version which we call the simple mixture. It remains still a very much open question if the $P2$-wave predicted by Biot's model may extend technical possibilities of the field measurements. The $P2$-wave itself is much too strongly attenuated to be observable under real in situ conditions but the Stoneley surface wave whose existence is due to the $P2$-wave gives rise to such new possibilities.

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