ON WAVES IN WEAKLY NONLINEAR POROELASTIC MATERIALS MODELING IMPACTS OF METEORITES

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dedicated to prof. Tommaso Ruggeri (Bologna)
on the occasion of his 60th Anniversary

Summary

We present a weakly nonlinear model of poroelastic materials in which deformations of both components are assumed to be small and sinultaneously material properties depend on the current porosity. Changes of the latter are described by the balance equation. The model is used in the description of nonlinear waves (one-dimensional Riemann problem) created by the impact of meteorite of a moderate size. The asymptotic method of analysis is applied.

1. Introduction

The paper is devoted to the construction of a weakly nonlinear model of a two-component porous material. It is assumed that deformations are small but material parameters depend on changes of porosity. The latter are described by a balance equation. This model is used in the description of propagation of strong discontinuity waves which may appear in soils after an impact of the meteorite of a moderate size. These are meteorites listed in the second and third lines of the Table 1 below.

With a few exceptions the propagation of shock waves in soils has been modelled by the application of one-component models (e.g. [1]). Some results based on the asymptotic analysis of a two-component model have been published under the smallness assumption of the porosity relaxation time (e.g. [3]). In this work, we present an approach in which this assumption is not satisfied.

2. Field equations

The fundamental fields descibing the mechanical behaviour of the porous material are as follows

– partial mass densities, ρ^S, ρ^f, of the soil and of the water, respectively,

– partial velocities, v_k^S, v_k^F, where we use Cartesian coordinates,

– porosity n.

In addition, as we construct the set of the first order field equations, we consider the deformation tensor of the solid component, e_{kl}^S, as the field as well.

These quatities have to satisfy the following set of balance equations

Table 1: *Consequences of impacts of meteorites of various sizes*

Impactor diameter	**Yield** (megatons)	**Interval** (years)	**Consequences**
<50	<10	<1	meteors in upper atmosphere most do not reach surface
75	10-100	1000	irons make craters like Barringer Meteor Crater (USA), stones produce airburst like Tunguska, land impacts destroy area of the size of cities
160	100-1000	5000	irons, stones hit the ground, comets produce airbursts, land impacts destroy area of the size of large urban area (New York, Tokyo)
350	1000-10000	15000	land impacts destroy area of the size of small states, ocean impacts produce mild trunamis
700	10000-100000	65000	land impacts destroy area of the size of moderate states like Virginia, ocean impacts makes big tsunamis
1700	100000-1000000	250000	land impact raises dust with global implication, it destroys area of the size of large states like California or France

$$\frac{\partial \rho^S}{\partial t} + \frac{\partial \rho^S v_k^S}{\partial x^k} = 0, \quad \frac{\partial \rho^F}{\partial t} + \frac{\partial \rho^F v_k^F}{\partial x^k} = 0,$$

$$\begin{aligned} \rho^S \left(\frac{\partial v_k^S}{\partial t} + v_l^S \frac{\partial v_k^S}{\partial x^l} \right) - \frac{\partial \sigma_{kl}^S}{\partial x^l} - \pi \left(v_k^F - v_k^S \right) &= 0, \\ \rho^F \left(\frac{\partial v_k^F}{\partial t} + v_l^F \frac{\partial v_k^F}{\partial x^l} \right) - \frac{\partial \sigma_{kl}^F}{\partial x^l} + \pi \left(v_k^F - v_k^S \right) &= 0, \end{aligned} \tag{1}$$

$$\frac{\partial \Delta_n}{\partial t} + v_k^S \frac{\partial \Delta_n}{\partial x^k} + \Phi \frac{\partial}{\partial x^k} \left(v_k^F - v_k^S \right) + \frac{\Delta_n}{\tau} = 0, \quad \Delta_n = n - n_E,$$

$$\frac{\partial e_{kl}^S}{\partial t} - \frac{1}{2} \left(\frac{\partial v_k^S}{\partial x_l} + \frac{\partial v_l^S}{\partial x_k} \right) = 0,$$

where we have used the assumption on smallness of deformation of both components

$$\begin{aligned} \max \left(\left\| e_{kl}^S \right\|, |\varepsilon| \right) &\ll 1, \quad \left\| e_{kl}^S \right\| = \sup_{\alpha=1,2,3} \left| \lambda^{(\alpha)} \right|, \quad \det \left(e_{kl}^S - \lambda^{(\alpha)} \delta_{kl} \right) = 0, \\ |\varepsilon| &\ll 1, \quad \varepsilon = \frac{\rho_0^F - \rho^F}{\rho_0^F}. \end{aligned} \tag{2}$$

The last equation in the set (1) is the linearized form of the integrability condition. We have used already the linear constitutive relations for the momentum source: $\pi \left(v_k^F - v_k^S \right)$, and for the porosity source: Δ_n / τ. In these relations π is the permeability coefficient, and τ is the relaxation time of porosity. The flux of porosity has been linearized as well: $\Phi \left(v_k^F - v_k^S \right)$, where Φ is the transport coefficient for porosity.

In order to construct field equations we have to add constitutive relations for the partial stresses, σ^S_{kl}, in the soil, the partial pressure in the fluid, p^F, and the equilibrium porosity, n_E. We assume them to have the following form

$$\begin{aligned}
\sigma^S_{kl} &= \sigma^S_{0kl} + \lambda^S(n)\, I\delta_{kl} + 2\mu^S(n)\, e^S_{kl} + \left[Q(n)\,\varepsilon - N(n)(n-n_0) - \beta\Delta_n\right]\delta_{kl}, \\
p^F &= p^F_0 - \rho^F_0 \kappa(n)\,\varepsilon - Q(n)\, I - N(n)(n-n_0) - \beta\Delta_n, \quad I = e^S_{mm}, \quad \sigma^F_{kl} = -p^F\delta_{kl}, \\
n_E &= n_0\left(1 + \delta(n)\, I\right).
\end{aligned} \tag{3}$$

Consequently, we have to specify the following material parameters

$$\left\{\lambda^S, \mu^S, \kappa, Q, N, \delta, \Phi, \beta, \pi, \tau\right\}. \tag{4}$$

They may all be functions of the current porosity, n, which makes the model weakly nonlinear.

In constitutive relations (3) we have included interactions between components through volume changes – the parameter Q introduced by M. Biot, as well as through the porosity changes – the parameter N, introduced on the basis of thermodynamical considerations [4]. However, we have neglected a second order contribution introduced by Signorini and relating stresses to the second invariants of deformations. This has been argued elsewhere to be negligible.

3. 1-D problem

In order to investigate the structure of the field equations we simplify the problem to one spacial dimension. In principle, a similar analysis can be also performed for three dimensions but the problem becomes technically much more involved.

The 1-D problem is described by the set of the following fields

$$\left\{e^S, \varepsilon, v^S, v^F, \Delta_n\right\}, \tag{5}$$

where e^S is the extension/compression of the skeleton (soil) in the x-direction, ε is the relative volume change of the fluid, v^S, v^F are x-components of the velocity, and $\Delta_n = n - n_E$ is the deviation of porosity from its equilibrium value.

Field equations for these quantities follow from (1), (3) and they have the form

$$\begin{gathered}
\frac{\partial e^S}{\partial t} - \frac{\partial v^S}{\partial x} = 0, \quad \frac{\partial \varepsilon}{\partial t} - \frac{\partial v^F}{\partial x} = 0, \\
\begin{aligned}
\rho^S\left(\frac{\partial v^S}{\partial t} + v^S\frac{\partial v^S}{\partial x}\right) - \frac{\partial \sigma^S}{\partial x} - \pi\left(v^F - v^S\right) &= 0, \\
\rho^F\left(\frac{\partial v^F}{\partial t} + v^F\frac{\partial v^F}{\partial x}\right) + \frac{\partial p^F}{\partial x} + \pi\left(v^F - v^S\right) &= 0,
\end{aligned} \\
\frac{\partial \Delta_n}{\partial t} + v^S\frac{\partial \Delta_n}{\partial x} + \Phi\frac{\partial}{\partial x}\left(v^F - v^S\right) + \frac{\Delta_n}{\tau} = 0,
\end{gathered} \tag{6}$$

with constitutive relations

$$\Delta_n = n - n_E, \quad n_E = n_0\left(1 + \delta e^S\right),$$

$$\begin{aligned}
\sigma^S &= \sigma^S_0 + \left(\lambda^S + 2\mu^S\right)e^S + Q\varepsilon - N\left(\Delta_n + \delta n_0 e^S\right) - \beta\Delta_n, \\
p^F &= p^F_0 - \rho^F_0\kappa\varepsilon - Qe^S - N\left(\Delta_n + \delta n_0 e^S\right) - \beta\Delta_n.
\end{aligned} \tag{7}$$

The dependence of compressibilities $\lambda^S + \frac{2}{3}\mu^S$, κ, coupling parameters Q, N, and properties δ, Φ of porosity equation can be found from Gedankenexperiments as functions of porosity (see: [4]). On the other hand, the nonequilibrium parameters π, τ, β are assumed in the present work to be constant. Their average values can be estimated (e.g. [5]) but little is known about their behaviour for large changes of porosity. In Table 2, we present these estimates of typical quantities in the case of soils saturated with water.
Bearing these estimates in mind, we can immediately find the orders of magnitude of various contributions to fields equations. These are presented in Table 3. It is seen that the model contains two small parameters: β and $1/\tau$. The last condition is different from that used in the work [3], where it was assumed that the relaxation time τ is small.

Table 2: *Typical values of parameters*

macroscopic time scale	$t_0 = 10^{-5}$ [s]
macroscopic length scale ($c = L/t_0 = 10^3$ m/s)	$L = 10^{-2}$ [m]
reference porosity, n_0	0.3
true mass density of solid: ρ_0^{SR} ($\rho_0^S = (1 - n_0)\, \rho_0^{SR}$)	$5 * 10^3$ [kg/m^3]
true mass density of fluid: ρ_0^{FR} ($\rho_0^F = n_0 \rho_0^{FR}$)	10^3 [kg/m^3]
effective Lamé coefficient, λ^S	50 [GPa]
coefficient of fluid compressibility, κ	10^6 [m^2/s^2]
coupling parameter, Q coupling parameter, N	0.2 [GPa] 0.2 [GPa]
coefficient of permeabilty, π	10^8 [kg/m^{3}s]
coefficient of equilibrium changes of porosity, δ	2 [-]
relaxation time of porosity, τ coefficient Φ	1 [s] 0.015 [-]
coupling coefficient, β	0.1 [GPa]
speeds of $P1-$ and $S-$waves	3 [km/s] and 1 [km/s]
changes of partial density, $\rho^F - \rho_0^F$	1 [kg/m^3]
diffusion (seepage) velocity, $v^F - v^S$	0.1 [m/s]
partial pressure in fluid, p_0^F partial stresses in solid, σ_0^{11}	1 [MPa] 150 [MPa]
changes of porosity, $n - n_0$	10^{-3}

Table 3: *Orders of magnitude of various contributions to field equations*

$[v^s] = 10$ [m/s]	$\left[v^F\right] = 10$ [m/s]
$\left[\rho^S \partial_t v^S\right] = 3 * 10^9$ [kg/m^{2}s]	$\left[\rho^F \partial_t v^F\right] = 10^9$ [kg/m^{2}s]
$\left[\rho^S v^S \partial_x v^S\right] = 3 * 10^8$ [kg/m^{2}s]	$\left[\rho^F v^F \partial_x v^F\right] = 10^8$ [kg/m^{2}s]
$\left[\sigma^S\right] = 150$ [MPa]	$\left[p^F\right] = 1$ [MPa]
$\left[\left(\lambda^S + 2\mu^S\right) e^S\right] = 150$ [MPa]	$\left[\rho_0^F \kappa \varepsilon\right] = 1$ [MPa]
$[Q\varepsilon] = 0.2$ [MPa]	$\left[Qe^S\right] = 0.2$ [MPa]
$[\Delta_n] = 1.5 * 10^{-5}$	$\partial_t \Delta_n = 1.5 * 10^0$ [1/s] $[\Delta_n/\tau] = 1.5 * 10^{-5}$ [1/s]
$\left[N\left(\Delta_n + \delta n_0 e^S\right)\right] = 0.12$ [MPa]	$[\beta \Delta_n] = 1.5 * 10^{-3}$ [MPa] $[\beta \partial_x \Delta_n] = 1.5 * 10^{-1}$ [kg/m^{2}s]
$\left[\partial_x \sigma^S\right] = 1.5 * 10^{10}$ [kg/m^{2}s]	$\left[\partial_x p^F\right] = 10^8$ [kg/m^{2}s]

In the dimensionless form, these parameters can be defined as follows
Estimates shown in Table 2 yield the following small parameters of the model

$$\varsigma^2 = \frac{\left[\frac{\Delta_n}{\tau}\right]}{\left[\frac{\partial \Delta_n}{\partial t}\right]} = 10^{-5}, \quad \beta' = \frac{\left[(N+\beta)\frac{\partial \Delta_n}{\partial x}\right]}{\left[\rho^F \frac{\partial v^F}{\partial t}\right]} = 0.45 * 10^{-2} \sim \varsigma. \tag{8}$$

Consequently, for our data we can redefine the field equations in terms of a single small parameter ς. They have the following form
– the equations resulting from the mass balance relations

$$\frac{\partial e_s}{\partial t'} - \frac{\partial v_s}{\partial x'} = 0, \quad \frac{\partial \varepsilon_f}{\partial t'} - \frac{\partial v_f}{\partial x'} = 0, \tag{9}$$

$$\begin{aligned} e_s &= \frac{e^S}{e_0}, \quad \varepsilon_f = \frac{\varepsilon}{e_0}, \quad e_0 = \frac{v_0 t_0}{L} = 1, \quad v_{s,f} = \frac{v^{S,F}}{v_0}, \\ v_0 &= \frac{L}{t_0}, \quad x' = \frac{x}{L}, \quad t' = \frac{t}{t_0}, \end{aligned} \tag{10}$$

– the equations resulting from momentum balance equations

$$\begin{aligned} &\rho_s \left(\frac{\partial v_s}{\partial t'} + v_s \frac{\partial v_s}{\partial x'}\right) - \frac{\partial \sigma_s}{\partial x'} - \pi' (v_f - v_s) = 0, \\ &\rho_f \left(\frac{\partial v_f}{\partial t'} + v_f \frac{\partial v_f}{\partial x'}\right) + \frac{\partial p_f}{\partial x'} + \pi' (v_f - v_s) = 0, \end{aligned} \tag{11}$$

$$\rho_s = \frac{\rho^S}{\rho_0}, \quad \rho^S = \rho_0^S \left(1 - e^S\right), \quad \rho_f = \frac{\rho^F}{\rho_0}, \quad \pi' = \frac{\pi}{\pi_0}, \quad \pi_0 = \frac{\rho_0}{t_0}, \tag{12}$$

– porosity balance equation

$$\frac{\partial \Delta'_n}{\partial t'} + v_s \frac{\partial \Delta'_n}{\partial x'} + \frac{\partial}{\partial x'}(v_f - v_s) + \varsigma^2 \Delta'_n = 0, \tag{13}$$

$$\Delta'_n = \frac{\Delta_n}{\Delta_0}, \quad \Delta_0 = \Phi, \quad \varsigma^2 = \frac{t_0}{\tau}, \tag{14}$$

where the dimensionless constitutive relations have the form

$$\begin{aligned} \sigma_s &= \sigma_{0s} + E' e_s + Q' \varepsilon_f - \varsigma \Delta'_n, \quad \sigma_s = \frac{\sigma^S}{\sigma_0}, \quad \sigma_{0s} = \frac{\sigma_0^S}{\sigma_0}, \quad \sigma_0 = \rho_0 \frac{L^2}{t_0^2}, \\ E &= \lambda^S + 2\mu^S - N\delta n_0, \quad E' = \frac{E}{\sigma_0}, \quad Q' = \frac{Q}{\sigma_0}, \end{aligned} \tag{15}$$

$$\begin{aligned} p_f &= p_{0f} - K' \varepsilon_f - C' e_s - \varsigma \Delta'_n, \quad p_f = \frac{p^F}{\sigma_0}, \quad p_{0f} = \frac{p_0^F}{\sigma_0}, \\ K &= \rho_0^F \kappa, \quad K' = \frac{K}{\sigma_0}, \quad C' = \frac{Q - N\delta n_0}{\sigma_0}. \end{aligned} \tag{16}$$

For the data of Tables 2 and 3 the parameters of the dimensionless quantities have the following values

$$\begin{aligned} L &= 10^{-2}\ [\mathrm{m}], \quad t_0 = 10^{-5}\ [\mathrm{s}], \quad \rho_0 = 10^3\ [\mathrm{kg/m^3}], \quad \Phi = 0.015, \\ v_0 &= 10^3\ [\mathrm{m/s}], \quad \sigma_0 = 1\ [\mathrm{GPa}], \quad \pi_0 = 10^8\ [\mathrm{kg/m^3 s}], \quad \Delta_0 = 1.5 * 10^{-2}, \\ \varsigma &= 3 * 10^{-3}. \end{aligned} \tag{17}$$

4. Governing set of equations

Now we are in the position to write the governing set of equations for the one-dimensional fields $\{e_s, \varepsilon_f, v_s, v_f, \Delta_n\}$. They have the following form

$$\frac{\partial \Delta_n}{\partial t} + v_s \frac{\partial \Delta_n}{\partial x} + \frac{\partial}{\partial x}(v_f - v_s) + \varsigma^2 \Delta_n = 0,$$

$$\begin{aligned} \rho_s \left(\frac{\partial v_s}{\partial t} + v_s \frac{\partial v_s}{\partial x} \right) - \frac{\partial}{\partial x}(E(n)\, e_s + Q(n)\, \varepsilon_f) + \varsigma \frac{\partial \Delta_n}{\partial x} - \pi (v_f - v_s) &= 0, \\ \rho_f \left(\frac{\partial v_f}{\partial t} + v_f \frac{\partial v_f}{\partial x} \right) - \frac{\partial}{\partial x}(K(n)\, \varepsilon_f + C(n)\, e_s) - \varsigma \frac{\partial \Delta_n}{\partial x} + \pi (v_f - v_s) &= 0, \end{aligned} \qquad (18)$$

$$\frac{\partial e_s}{\partial t} - \frac{\partial v_s}{\partial x} = 0, \quad \frac{\partial \varepsilon_f}{\partial t} - \frac{\partial v_f}{\partial x} = 0,$$

where

$$\begin{aligned} \rho_s &= \rho_{0s}(1 - e_s), \quad \rho_f = \rho_{0f}(1 - \varepsilon_f), \\ \frac{dE}{dn} &< 0, \quad \frac{d^2E}{dn^2} > 0, \quad \frac{dK}{dn} < 0, \quad \frac{d^2K}{dn^2} > 0, \\ \frac{dQ}{dn} &> 0, \quad \frac{d^2Q}{dn^2} < 0, \quad \frac{dC}{dn} > 0, \quad \frac{d^2C}{dn^2} < 0, \end{aligned} \qquad (19)$$

and ρ_{0s}, ρ_{0f} are initial values of dimensionless partial mass densities.
For typographical reasons, we have skipped the prime for the dimensionless quantities. The inequalities (19) follow from micro/macro considerations presented in the paper [4]. The structure of this set makes clear that, for reasons of consistency of orders of magnitude, a weak discontinuity wave P1 or P2, i.e. for waves in which velocities are continuous on the wave front but one of the accelerations may be not, is accompanied by a kink-like solution for the porosity. This follows easily from momentum balance equations. Simultaneously, for strong discontinuities of velocity fields (i.e. kink-like solutions) the porosity must be expressed by the soliton-like solution. This structure has been already indicated for another asymptotic problem in the paper [3].

5. Structure of asymptotic solutions

We construct the asymptotic solution of the Riemann problem. Technical details concerning the asymptotic analysis can be found in the papers [2]. In such a solution, we have a regular part which is a power expansion with respect to the small parameter ς and the singular part related to the "boundary layer" effect near the front of the wave. This contribution is written in the form of dependence on the so-called fast variable $\sigma = \left(x - \overset{0}{x}(t) \right) / \varsigma^2$, where $\overset{0}{x}(t)$ is the position of the wave front. For a typical quantity Π this expression have the following form

$$\Pi_{as} = \Gamma_0(t) + \Pi_0(\sigma, t) + \sum_{j=1}^{N} \varsigma^j \left(\Gamma_1(x, t) + \Gamma_j^p(\sigma, x, t) \right),$$

$$\begin{aligned} \Gamma_j^p(\sigma, x, t) &= \Pi_j(\sigma, t) + H_j^p z_0(\sigma, t), \quad j \geq 1, \\ \overset{0}{x}(t) &= X_0(t) - \varsigma X_1(t) - \varsigma^2 X_2(t), \end{aligned} \qquad (20)$$

where $\Gamma_j, \Pi_j H_j^p, z_0$ are smooth bounded functions, and, simultaneously, Π_j is kink-like, while z_0 – soliton-like. They are stabilized in infinity, i.e.

$$\begin{aligned}
\forall k, i, j, l \quad &: \quad \sigma^k \frac{d^j}{d\sigma^j} \frac{d^l}{dt^l} \frac{d^i}{dx^i} \left(y - y^{\pm}\right) = 0 \text{ if } \sigma \to \infty, \\
z_0^- \quad &= \quad \lim_{\sigma \to -\infty} z_0 = 0, \quad z_0^+ = \lim_{\sigma \to \infty} z_0 = 0, \text{ or conversely,} \\
\Pi_j^{\pm} \quad &= \quad 0 \text{ (solitons)},
\end{aligned} \tag{21}$$

where y is either z_0 or Π_j. Then
Proposition 1: *Strong discontinuities cannot exist simultaneously for v_s and v_f, i.e. the strong discontinuity of v_s yields the weak discontinuity of v_f, or conversely. The propagation velocities of fronts of these two discontinuities satisfy the condition*

$$\dot{X}_0^2 = \frac{E\left(n\right)}{\rho_{0s}} \text{ or } \dot{X}_0^2 = \kappa\left(n\right). \tag{22}$$

Proposition 2: *Let $\Delta_n = v_s = v_f = 0$ and $\rho_{0s} > 0, \rho_{0f} > 0, e_s$ are constant for $t = 0$. Then asymptotic solutions of any accuracy with respect to ς of the system* (18) *exist on a finite time interval, and they possess the following properties. They are smooth approximations with respect to ς of order $O\left(\varsigma\right)$ of strong discontinuities either for v_s, ρ_s, e_s or for $v_f, \rho_f, \varepsilon_f$ and the infinitely thin soliton function of order $O\left(\varsigma\right)$ for Δ_n along a small perturbation of characteristics of the linearized problem to* (6).
Details of these asymptotic considerations are the subject of the forthcoming paper.

6. Final remarks

The model presented in this work is able to describe the propagation of strong discontinuity waves which propagate in the intermediate stage of the impact problem. It enables the estimation of the time and the distance from the site of impact before the dynamics of the problem is described by the usual acoustic waves in the two-component poroelastic medium.

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